CO781/QIC890 Assignment 1

Due Friday night Oct 09, 2020, 10:00pm

Instruction: Please submit your solutions to Crowdmark by the due date and time. Take special care to place the answer to each question in the right place.

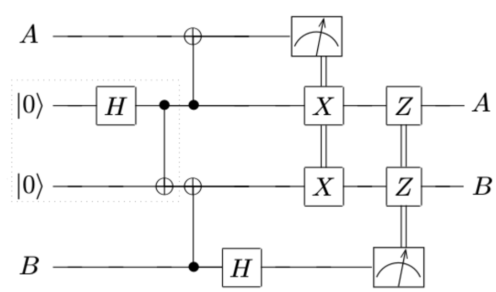
Question 1. Resource inequalities concerning the CNOT [11 marks]

You can use any resource inequality proved in class (but please state it clearly before use).

Define CNOT acting on AB as the gate taking $a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \to a|00\rangle + b|01\rangle + c|11\rangle + d|10\rangle$ where the first and second qubits are systems A and B respectively. Suppose two remote parties Alice and Bob are in possession of A and B respectively.

(a) [3 marks] Show that 1 $\text{cbit}_\to$ + 1 $\text{cbit}_\leftarrow$ + 1 ebit $\geq$ CNOT. That is, Alice and Bob can perform a CNOT on AB using one maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, one classical bit of communication from Alice to Bob, and one classical bit of communication from Bob to Alice. To do so,

(i) verify that the following circuit implements the desirable CNOT.



In the above, circuit convention follows Nielsen and Chuang. In particular, a vertical *double line* connecting a measurement box (in the computational basis) to a unitary U means that "conditioned on the measurement outcome being 1, perform U, otherwise do nothing". X, Z, H are the Pauli-x, Pauli-z, and the Hadamard gates.

(ii) Derive the claimed resource inequality from the circuit.

(b) [2 mark] Use part (a) to show that 1 $\text{cobit}_\to$ + 1 $\text{cobit}_\leftarrow$ $\geq$ CNOT + 1 ebit.

(c) [1 mark] Show that CNOT + 1 ebit $\geq$ 1 $\text{cbit}_\to$ + 1 $\text{cbit}_\leftarrow$.

Hint: Suppose Alice wants to send $a \in 0, 1$ to Bob, and Bob wants to send $b \in 0, 1$ to Alice. How can they jointly transform the share ebit to one out of 4 possible Bell states? How can they use the CNOT (together with local operations) so that the final output is $|b\rangle_A|a\rangle_B$?

(d) [1 mark] Use part (c) to show that CNOT + 1 ebit $\geq$ 1 $\text{cobit}_\to$ + 1 $\text{cobit}_\leftarrow$.

(e) [1 mark] Show that 2 CNOT = SWAP.

(f) [3 marks] We say that the rate pair (r, s) is achievable if there is a protocol communicating rn bits from Alice to Bob, and sn bits from Bob to Alice, using n CNOTs and unlimited number of ebits. What is the set of all achievable rate pairs? (You need to show that all points in the region are achievable, and all points outside are not. Please use parts (a) and (c) but please do not use part (e).)

Question 2. Superdense coding of quantum states [10 marks]

Let A and B be quantum systems of dimensions d_A and d_B respectively, for $d_A \geq d_B \geq 3$. A state $|\Psi\rangle \in A \otimes B$ is called maximally entangled if $\mathrm{tr}_A |\Psi\rangle\langle\Psi| = \frac{I}{d_B}$.

You can assume the following fact from quant-ph/0407049. (Sec VII in the paper has a very brief description of the solution, which you can consult. Thoughts on the questions have been discussed at the end of lecture 6. You should write up your own, detailed, solution.)

Let $0 < \alpha < \log d_B$, $\beta = \frac{1}{\ln 2} \frac{d_B}{d_A}$, and Γ be an absolute constant which may be chosen to be $1/1753$. There exists a subspace $S \subset A \otimes B$ of dimension

$$s = \left\lfloor d_A d_B \frac{\Gamma \alpha^{2.5}}{(\log d_B)^{2.5}} \right\rfloor$$

with the property that for every $|\psi\rangle \in S$, there exists a maximally entangled state $|\Psi\rangle \in A \otimes B$ such that

$$\| |\psi\rangle\langle\psi| - |\Psi\rangle\langle\Psi| \|_1 \leq \sqrt[4]{16(\alpha + \beta)}.$$

You can also use without proof that for any two maximally entangled states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ in $A \otimes B$, there exists a unitary $U \in \mathbb{U}(A)$ such that $|\Psi_2\rangle = (U \otimes I)|\Psi_1\rangle$

Your task: give a protocol so that for any $|\phi\rangle \in C^{d^2}$ such that Alice has a classical description of, she can prepare ρ_ϕ in Bob's laboratory given $\log d$ ebits and $\log d + 2.5 \log \log d - \log(\Gamma \alpha^{2.5})$ qbits$_\rightarrow$ (forward communication of qubits), where

$$\| |\phi\rangle\langle\phi| - \rho_\phi \|_1 \leq 2\sqrt[4]{2\alpha}.$$

Hints: you should set $d_B = d$ and $d_A = \frac{d(\log d)^{2.5}}{\Gamma \alpha^{2.5}}$ so that the subspace $S \subset A \otimes B$ has d^2 dimensions. Note that $S \neq C^{d^2}$ but you can define a bijection between them, and you can assume that Alice and Bob pre-agree on the bijection. You should handle all the approximations carefully.