Quantum Error Correction and Fault Tolerance, Winter 2022

Problem Set 3

Due: March 7 8pm on Crowdmark

1. Hypergraph product codes [11 marks]

Let C_1 be the [7,4,3] Hamming code, whose parity-check matrix can be written

$$H_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and let C_2 be the [3,1,3] repetition code, whose parity-check matrix can be written

$$H_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) Compute the parameters (n, k, and d) of the transpose codes C_1^T and C_2^T, where we recall that the transpose code of a code C with parity-check matrix H is the code with parity-check matrix H^T. **[2 marks]**

Hint: if $k^T = 0$ then we define $d^T = \infty$.

(b) Compute the parameters of the hypergraph product code $\mathrm{HGP}(H_1, H_2)$. **[1 mark]**

Let

$$H_1' = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

and let

$$H_2' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

These are overcomplete parity-check matrices of C_1 and C_2, respectively.

(c) Show that the hypergraph product code $\mathrm{HGP}(H_1', H_2')$ has parameters $n = 33, \quad k = 5, \quad d = 3$. **[5 marks]**

(d) Show that

$$v = [0,0,0,0,0,1,0,0,1,0,0,1,0,0,1,0,0,0,0,0,0 \mid 0_{12}]^T$$

is a logical operator of $\mathrm{HGP}(H_1', H_2')$. **[3 marks]**

Hint: you may use the fact that any vector of the form $[x \otimes y \mid 0_{m_a m_b}]^T$, where $x \in \ker H_a, y \in (\operatorname{im} H_b^T)^\bullet$ is a logical X operator of $\mathrm{HGP}(H_a, H_b)$. We recall that the $\operatorname{im} H$ is the image (rowspace) of H. Note that given a vector space $V \subseteq \mathbb{F}_2^n$ its complement is a vector space $V^\bullet \subseteq \mathbb{F}_2^n$ such that $V \oplus V^\bullet = \mathbb{F}_2^n$.

2. Subsystem codes [11 marks]

Consider the following (quantum) Tanner graph. Recall that circles represent qubits, white squares represent X checks, black squares represent Z checks.

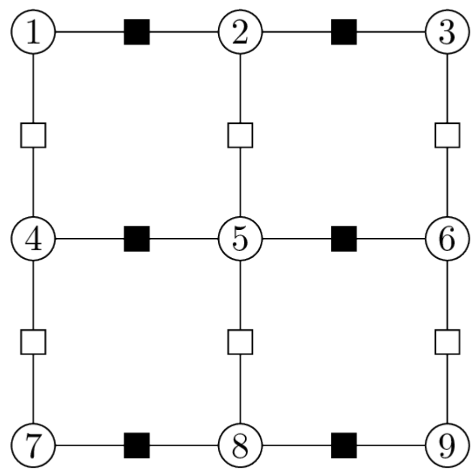

Figure 1: Bacon-Shor code Tanner graph.

(a) Write down the X parity-check matrix, H_X, for this Tanner graph, with the column order matching the indexing of the qubits given in Figure 1. **[1 mark]**

(b) Write down the Z parity-check matrix, H_Z, for this Tanner graph, with the column order matching the indexing of the qubits given in Figure 1. **[1 mark]**

Observe that checks do not commute, i.e., $H_X H_Z^T \neq 0$, and therefore these matrices do not define a CSS stabilizer code. There exists a generalization of stabilizer codes called subsystem codes, which are defined by non-commuting checks. In a (CSS) subsystem code, the X stabilizers are the elements in the rowspace of H_X that commute with the Z checks, i.e., have even overlap with all the rows of H_Z. Analogously, the Z stabilizers are the elements in the rowspace of H_Z that commute with the X checks.

(c) Write down generating sets, S_X and S_Z, for the X stabilizers and the Z stabilizers for the subsystem code defined by H_X and H_Z. **[4 marks]**

(d) The formula for the number of encoded qubits in a CSS subsystem code is

$$k = n - (1/2)(\operatorname{rank} H_X + \operatorname{rank} S_X + \operatorname{rank} H_Z + \operatorname{rank} S_Z)$$

Calculate the number of encoded qubits in the subsystem code defined by H_X and H_Z. **[1 mark]**

(e) A logical Z operator of a subsystem code is an operator that is in $\ker S_X$ (commutes with the X stabilizers) but is not in $\operatorname{im} H_Z^T$ (the rowspace of the Z checks). Give an example of a logical Z operator of the subsystem code defined by H_X and H_Z, and justify your choice. **[4 marks]**

3. Distance balancing [5 marks]

Consider the following generalization of the hypergraph product. Let $\mathcal{Q}$ be an $[[n, k, d]]$ CSS stabilizer code with X and Z parity-check matrices $H_X \in \mathcal{M}_{m_x \times n}(\mathbb{F}_2)$ and $H_Z \in \mathcal{M}_{m_z \times n}(\mathbb{F}_2)$, i.e., the code has m_X X stabilizer generators and m_Z Z stabilizer generators. Let d_X and d_Z denote the X and Z distances of $\mathcal{Q}$. Let $\mathcal{C}$ be an $[n', k', d']$ linear code with parity-check matrix $H \in \mathcal{M}_{m \times n}$. The generalized hypergraph product of $\mathcal{Q}$ and $\mathcal{C}$ is a CSS stabilizer code with parameters

$$N = nn' + m_Z m', \quad K = kk', \quad D_X = d_X d', \quad D_Z = d_Z.$$

(a) Suppose that a family of CSS codes $\{\mathcal{Q}\}$ has parameters

$$n, \quad k = \sqrt{n}, \quad d_X = \sqrt{n}, \quad d_Z = n^{3/4},$$

and $\mathcal{C}$ is a classical repetition code with $n' = n^{1/4}$. Write down the parameters of the code family constructed via the generalized hypergraph product of $\mathcal{Q}$ and $\mathcal{C}$, expressed as a function of N. **[4 marks]**

Hint: one can always find a set of Z stabilizer generators H_Z with $m_Z = O(n)$.

(b) Now replace $\mathcal{C}$ in (a) with a good LDPC code with parameters $[n', \Theta(n'), \Theta(n')]$, where $n' = n^{1/4}$. Write down the parameters of the code family constructed via the generalized hypergraph product of $\mathcal{Q}$ and $\mathcal{C}$, expressed as a function of N. **[1 mark]**