Quantum Error Correction and Fault Tolerance, Winter 2024

Problem Set 4

Due: Friday March 22, 2024

1. Flag error correction [8 marks]

(a) The 5-qubit code has stabilizer by $XZZXI$ and its cyclic permutations. Its logical operators are $\overline{X} = XXXXX$ and $\overline{Z} = ZZZZZ$. The naive circuit for measuring the stabilizer $XZZXI$ is shown in Figure 1. Assume that the first CZ is followed by one of XI, IX, XX, ZI, IZ ZZ, IY, YI, YY, XZ, ZX, XY, YX, ZY, YZ, chosen at random, acting on the two qubits in the support of the first CZ. List all of the 'bad' faults that can lead to a multi-qubit data error, and their corresponding data error. [**4 marks**]

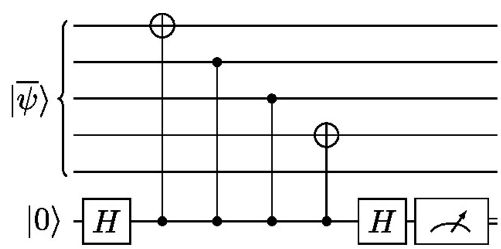

Figure 1: Naive 5-qubit code stabilizer measurement circuit

(b) The circuit shown in Figure 2 incorporates a flag qubit to catch the bad faults. For each bad fault identified above, verify that the measurement of the flag qubit yields -1, heralding the bad fault. [**4 marks**]

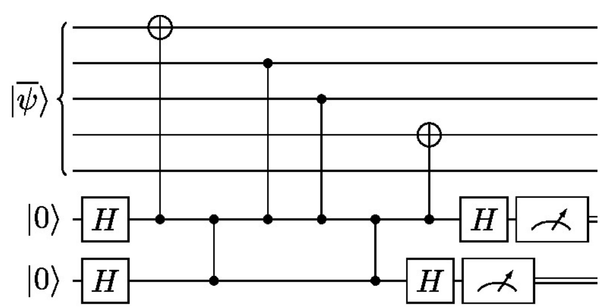

Figure 2: Flag 5-qubit code stabilizer measurement circuit

2. Steane error correction [12 marks]

Suppose we encode a qubit in a $k = 1$ CSS code. In Steane error correction we use logical $|\overline{0}\rangle$ and $|\overline{+}\rangle$ ancillas, encoded in the same code as the data. We can extract all the stabilizer eigenvalues simultaneously using the circuit in Figure 3. The X stabilizers of a CSS code define a classical code. To find the recovery operator in Steane error correction, we simply do classical error correction on the ancilla measurement outcomes.

(a) Recall that Steane's code is a [[7,1,3]] code with stabilizer generators $X_1X_2X_4X_5$, $X_2X_3X_4X_6$, $X_4X_5X_6X_7$, $Z_1Z_2Z_4Z_5$, $Z_2Z_3Z_4Z_6$, $Z_4Z_5Z_6Z_7$, and logical operators $\overline{X} = X^{\otimes 7}$ and $\overline{Z} = Z^{\otimes 7}$. Write down the logical $|\overline{0}\rangle$ and $|\overline{+}\rangle$ states for this code. **[2 marks]**

(b) Suppose there is an X_1Z_4 error on the data. Apply Steane error correction and write down the error syndrome (the list of stabilizer generators with -1 eigenvalue). What is the minimum weight recovery operator? **[3 marks]**

(c) Suppose there is an X_3X_4 error on the data and an X_1Z_7 error on the $|\overline{+}\rangle$ ancilla. Apply Steane error correction and compute the error syndrome and the minimum weight recovery operator. **[2 marks]**

(d) Let p be the probability of any fault in the Steane stabilizer measurement circuit. Assume a depolarizing noise model, i.e. each single-qubit gate is replaced by the gate followed by the depolarizing channel $\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$. Similarly, there is a depolarizing channel before each measurement. And each two-qubit gate is replaced by the ideal gate followed by a two-qubit depolaring channel, i.e. with probability $(1-p)$ apply II, and with probability p apply one of XI, IX, XX, ZI, IZ ZZ, IY, YI, YY, ZX, XZ, XY, YX, ZY, YZ, chosen at random.
Prove that Steane error correction succeeds with probability $1 - O(p^2)$. Success here means that performing ideal error correction on the output state state gives the same result as applying ideal error correction on the input state. You can assume that recovery operators are implemented perfectly (in real processors they are tracked in software). **[5 marks]**

Hint: You need to show that Steane error correction succeeds for each error occurring with probability p.

3. Knill error correction [12 marks]

Knill error correction is also known as error-correction via teleportation. It takes the form of a logical teleportation circuit, as shown in Figure 4.

We require a logical Bell state ancilla, $|\overline{00}\rangle + |\overline{11}\rangle$, encoded in two blocks of the same code as the data. For simplicity, assume that the data is a single logical qubit encoded in a $k = 1$ stabilizer code. We perform a transversal bell basis measurement between the data block and the first ancilla block. We break the bell basis measurement into two steps: a transversal CNOT followed by

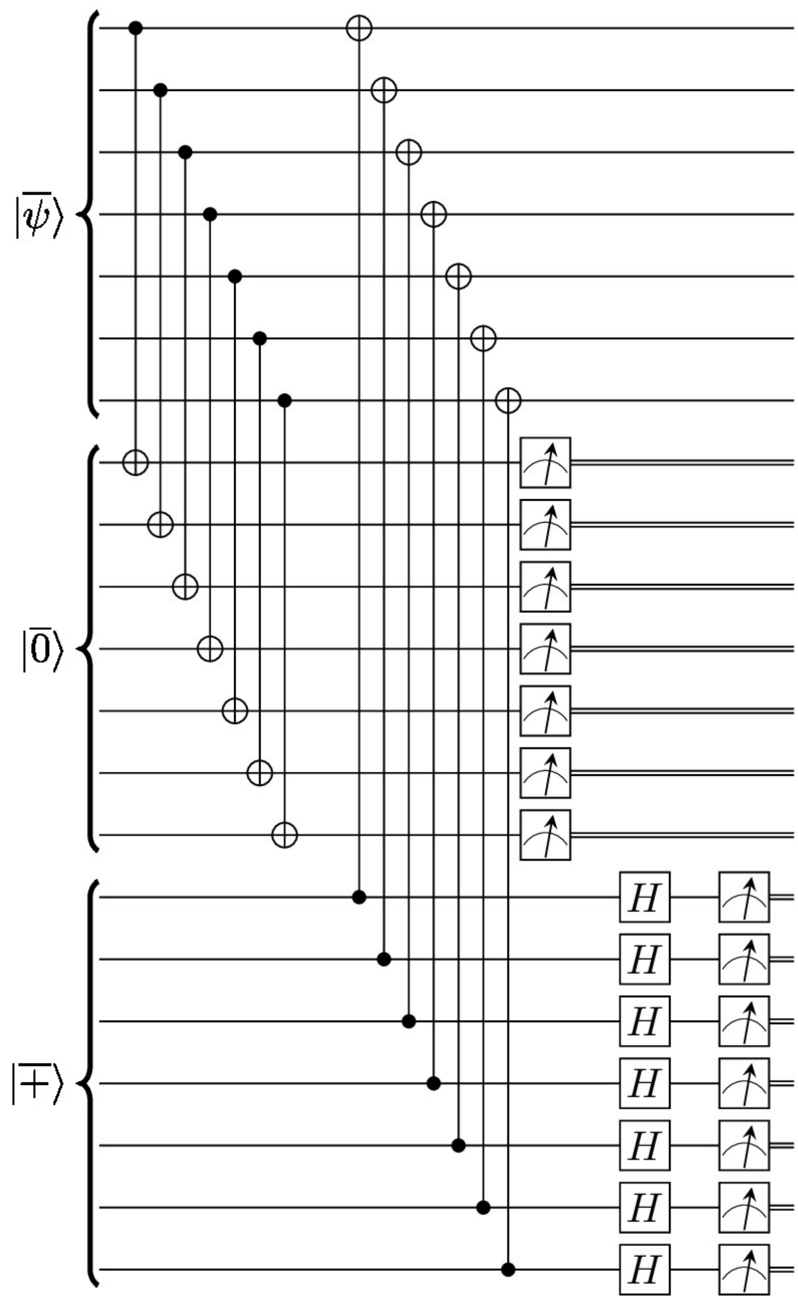

Figure 3: Steane error correction circuit

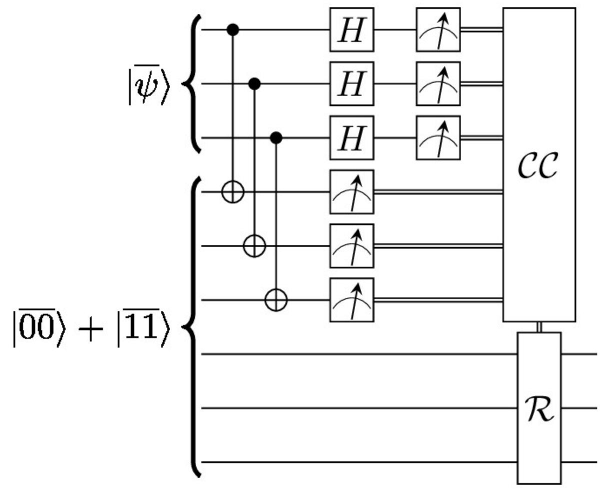

Figure 4: Knill error correction circuit, CC represents classical processing and R is the recovery operator. We show an example for a code with three physical qubits but the method works for any stabilizer code.

measuring all the data block qubits in the X basis and measuring all the qubits in the first ancilla block in the Z basis.

(a) Suppose an arbitrary pauli P acts on the data block and the same Pauli acting on the first ancilla block, possibly with a different sign. Compute the transformation of this operator under the action of the transversal CNOT. **[3 marks]**

Hint: write the operator as $(-1)^{b_j} P = i^{c(P_X, P_Z)}(-1)^{b_j} P_X P_Z$, where b_j indicates the sign of P on block j, P_X is a tensor product of X and I, P_Z is a tensor product of Z and I, and $c(A, B) = 0$ if A and B commute and $c(A, B) = 1$ if A and B anticommute.

The calculation in part (a) applies to *any* Pauli operator P, including the stabilizers of the code and the logical operators of the code. Therefore we can deduce the eigenvalues of the stabilizer generators $g_i \otimes g_i$ and the logical operators $\overline{X} \otimes \overline{X}$ and $\overline{Z} \otimes \overline{Z}$ by combining the measurement results for the qubits in the support of the relevant operator. This allows us to perform a logical teleportation of the data and error correction at the same time! Let's consider an example: the three-qubit repetition code. Recall that this code has stabilizer $\langle Z_1 Z_2, Z_2 Z_3 \rangle$ and logical operators $\overline{X} = X_1 X_2 X_2$ and $\overline{Z} = Z_1$. An arbitrary encoded state can be written as $|\overline{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle$.

(b) Starting from the state $|\overline{\psi}\rangle_1 \otimes (|\overline{00}\rangle_{23} + |\overline{11}\rangle_{23})$ compute the state after applying the transversal CNOT from block 1 to 2. **[1 mark]**

(c) Suppose the result of the Z basis measurement of block 2 is $|111\rangle$ and the result of the X basis measurement of block 1 is $|{+}{+}{-}\rangle$. What operators do we need to apply to correctly teleport the state? **[2 marks]**

(d) Now suppose that we start with the state $XII|\overline{\psi}\rangle_1 \otimes (|\overline{00}\rangle_{23} + |\overline{11}\rangle_{23})$, i.e. there is an X error on the data. Supposing that we measure $|011\rangle$ in block 2 and $|{+}{+}{+}\rangle$ in block 1, What operators do we need to apply to correctly teleport the state? **[3 marks]**

Hint: use the error syndrome (stabilizer eigenvalues) to deduce the correct values for the logical operators.

(e) Now consider the case where we have an IXI error on block 2 after the CNOT gates but before the measurements. Supposing that we measure $|010\rangle$ in block 2 and $|{-}{-}{-}\rangle$ in block 1, what operators do we need to apply to correctly teleport the state? **[3 marks]**