Problem Set 7

Quantum Error Correction, 2022 spring
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Problem 1. Hypercube code and multi-qubit control-Z gates (30)

This problem is a simpler version of section 4 of arXiv:1503.02065. Consider the following single-qubit phase operator:

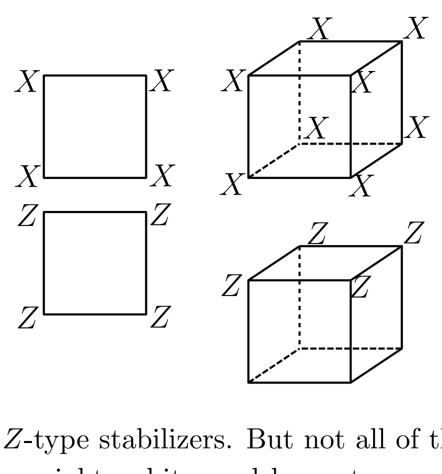
$$\mathcal{R}_m = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{2\pi}{2^m}} \end{bmatrix} = |0\rangle\langle 0| + e^{i\frac{2\pi}{2^m}}|1\rangle\langle 1| \qquad m \text{ is a non-negative integer.} \tag{1}$$

Here $\mathcal{R}_0 = I$, $\mathcal{R}_1 = Z$, $\mathcal{R}_2 = S$ and $\mathcal{R}_3 = T$. In the lecture, we learned that the D-dimensional topological color code with certain boundaries has a single logical qubit ($k = 1$) with the following transversal logical operator:

$$\overline{\mathcal{R}_D} = (\mathcal{R}_D)^{\otimes n_{\text{odd}}} \otimes (\mathcal{R}_D)^{\otimes n_{\text{even}}} \tag{2}$$

where n_{odd} and n_{even} are the number of qubits at odd and even sites when the lattice is viewed as a bipartite graph. Namely, we showed that $\overline{\mathcal{R}_D}$ acts as a logical $\mathcal{R}_D$ (or $\mathcal{R}_D^\dagger$) operator. We also learned that the smallest realization is the so-called D-th level Reed-Muller code.

In this problem, we treat the cases where the color code has multiple logical qubits. The code below is the smallest realization of the D-dimensional topological color code with $k = D$ logical qubits. Consider a stabilizer code defined on a d-dimensional hypercube with $n = 2^D$ qubits living on vertices. The code has only one X-type stabilizer generator, $X^{\otimes n}$, acting on all the qubits, while Z-type stabilizer generators are four-body and are defined on each two-dimensional face. Two-dimensional and three-dimensional examples are shown below:



(3)

In three dimensions, there are six Z-type stabilizers. But not all of them are independent!

(The three-dimensional code has eight qubits, and has a transversal non-Clifford gate as we show below. To the best of my knowledge, this is the smallest qubit stabilizer code with such a property).

(a) Let us define a *commutator* of two unitary operators as follows:

$$\mathcal{K}(V,W) = VWV^{\dagger}W^{\dagger}. \tag{4}$$

Show that

$$\mathcal{K}(\mathcal{R}_m, X) \propto \mathcal{R}_{m-1} \qquad \text{for all } m \geq 1. \tag{5}$$

(b) Consider a Hilbert space of m qubits. Let X_1 be a Pauli-X acting on the first qubit. Let us define a multi-qubit Control-Z gate as follows:

$$\mathrm{C}^{\otimes m-1}Z|j_1,\ldots,j_d\rangle = (-1)^{j_1\cdots j_m}|j_1,\ldots,j_d\rangle \qquad j_m = 0,1. \tag{6}$$

Here $j_1\cdots j_m$ means a product of $j_1,\ldots,j_m$. Compute the commutator $\mathcal{K}(\mathrm{C}^{\otimes m-1}Z, X_1)$.

(c) Show that the code has D logical qubits. Show that the code distance (minimal weight of a non-trivial logical operator) is two.

(d) Show that $\overline{\mathcal{R}_d} = (R_d)^{\otimes n}$ is a logical operator of the code. Also show that it acts as a logical $\mathrm{C}^{\otimes d-1}Z$ gate. If you find this problem difficult, you can do the $D=3$ case only.

Problem 2. Price of a stabilizer code (30)

The price p of a stabilizer code is the volume of the smallest subsystem of qubits which supports all the logical operators.

(a) Find the price of the toric code defined with $N = 2L^2$ qubits.

(b) Find the price of the 15-qubit code.

(c) Prove that $p \leq n-d+1$ and $p \geq k+d-1$. (Hint: use the duality relation for the first inequality, and use the argument for proving the quantum Singleton bound for the second inequality).

Problem 3. Bound on local classical codes (20)

Consider a classical stabilizer code in D dimensions. Show that

$$kd^{\frac{1}{D-1}} \leq O(n). \tag{7}$$

Here stabilizer generators are tensor products of Pauli-Z operators, and the classical code distance d is the smallest subsystem of "qubits" which supports all the X-type logical operators.

Problem 4. Symmetry in a stabilizer code (20)

Consider a stabilizer code with $k=1$ defined on a D-dimensional hypercubic lattice ($N = L^D$). Assume that the stabilizer group $\mathcal{S}$ is invariant under finite translations:

$$T_1^{c_1}(\mathcal{S}) = \cdots = T_D^{c_D}(\mathcal{S}) = \mathcal{S} \tag{8}$$

where T_j are operators that shift qubits in the direction of $\hat{j}$. Here c_j are $O(1)$ constants. Let d_X, d_Z be the sizes of the smallest subsystem of qubits which support a logical-X and logical-Z operators respectively. Show that

$$d_X d_Z \geq O(N). \tag{9}$$

(Hint: We do not need to assume locality of stabilizer generators in this problem).