Definition:

Let $\Lambda = \sum_x p_x \, |x\rangle\langle x|_X \otimes \rho_{x\,Q}$.

$\mathcal{M}$ measurement on Q with output space Y

The accessible information for ensemble $\mathcal{E} = \{p_x, \rho_x\}$ is

$$I_{acc}(\mathcal{E}) := \max_{\mathcal{M}} I(X:Y)_{I \otimes \mathcal{M}(\Lambda)}$$

Example 1. $x = 0, 1,\ p(0) = p(1) = 1/2,$

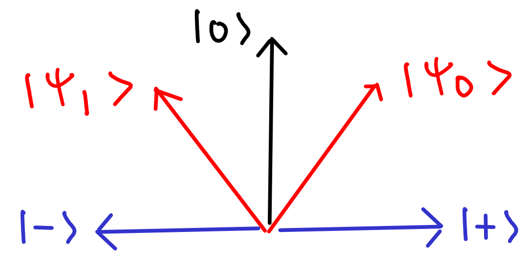
$a, b \geq 0\ ,\ a^2 + b^2 = 1$

$\rho_x = |\psi_x\rangle\langle\psi_x|\ ,\quad |\psi_0\rangle = a|0\rangle + b|1\rangle$

$|\psi_1\rangle = a|0\rangle - b|1\rangle$

(most general form of 2 arbitrary pure states)

Optimal measurement:
projective, along basis $\{|+\rangle, |-\rangle\}$



Levitin 95, or Fuchs PhD thesis 96 (Ch3.5)

$p(x=0\ y=+) = p(x=0)\ p(y=+|x=0)$

$p(x=0) = \frac{1}{2}$, $\quad p(y=+|x=0) = \mathrm{tr}\, |+\rangle\langle+|\psi_0\rangle\langle\psi_0| = \left(\frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}}\right)^2 = \frac{1}{2}(a^2+b^2+2ab) = \frac{1}{2} + ab$

$p(x=1\ y=+) = \frac{1}{2}\left(\frac{1}{2} - ab\right)$

$p(y=+) = 1/2,$

$p(x=0|y=+) = 1/2 + ab$
$p(x=1|y=+) = 1/2 - ab$

$H(X|y=+) = h(1/2+ab)$

Similarly,
$p(y= -) = 1/2,$
$H(X|y= -) = h(1/2-ab) = h(1/2+ab)$

$\therefore\ H(X|Y) = p(y=+)\, H(X|y=+) + p(y=-)\, H(X|y=-) = h(1/2+ab)$

$I_{acc} = I(X:Y) = H(X) - H(X|Y) = 1 - h(1/2+ab)$

How to optimize measurement for Iacc?

1. Unknown for most ensembles

   For the few ensembles (highly symmetric) with known optimal measurements, there is no simple proof of optimality :(

How to optimize measurement for Iacc?

2. EB Davies, IEEE Trans Info Th, 24, p596, 1978

For any ensemble of states in d dimensions, $\mathcal{E} = \{p_x, \rho_x\}$
optimal measurement has POVM $\mathcal{M} = \{M_y\}_{y=1}^{n}$ with

(a) rank(M_y) = 1 and

(b) $d \leq n \leq d^2$

Proof (a): If $M_y = \sum_k M_{y,k}$ is a decomp into rank 1 matrices

replace measurement $\mathcal{M}$ with POVM $\{M_y\}$

by new measurement $\mathcal{M}'$ with POVM $\{M_{y,k}\}$.

outcome
has 2 parts

eg $\mathcal{M}$: $M_0 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{3}|+\rangle\langle +|$ $(y=0)$

$M_1 = \frac{1}{5}|1\rangle\langle 1| + \frac{1}{2}|-\rangle\langle -|$ $(y=1)$

$M_2 = \frac{4}{5}|1\rangle\langle 1| + \frac{1}{6}|+\rangle\langle +|$ $(y=2)$

$\mathcal{M}'$: $M_{0,0} = \frac{1}{2}|0\rangle\langle 0|$ $(y=0, k=0)$ $\quad M_{0,1} = \frac{1}{3}|+\rangle\langle +|$ $(y=0, k=1)$

$M_{1,0} = \frac{1}{5}|1\rangle\langle 1|$ $(y=1, k=0)$ $\quad M_{1,1} = \frac{1}{2}|-\rangle\langle -|$ $(y=1, k=1)$

$M_{2,0} = \frac{4}{5}|1\rangle\langle 1|$ $(y=2, k=0)$ $\quad M_{2,1} = \frac{1}{6}|+\rangle\langle +|$ $(y=2, k=1)$

How to optimize measurement for Iacc?

2. EB Davies, IEEE Trans Info Th, 24, p596, 1978

For any ensemble of states in d dimensions, $\mathcal{E} = \{p_x, \rho_x\}$
optimal measurement has POVM $\mathcal{M} = \{M_y\}_{y=1}^{n}$ with

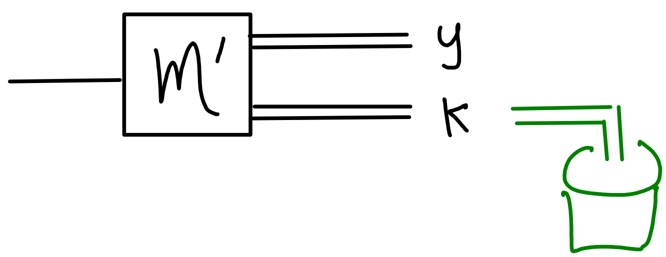
(a) rank(M_y) = 1 and

(b) $d \leq n \leq d^2$

Proof (a): If $M_y = \sum_k M_{y,k}$ is a decomp into rank 1 matrices

replace measurement $\mathcal{M}$ with POVM $\{M_y\}$
by new measurement $\mathcal{M}'$ with POVM $\{M_{y,k}\}$.

outcome has 2 parts



if we discard k, we perform $\mathcal{M}$

$$\therefore \; I_{acc}(\mathcal{E}) \underset{\substack{\uparrow \\ \text{given} \\ \mathcal{M}\text{ optimal}}}{=} I(X{:}Y)_{\mathcal{I}\otimes\mathcal{M}(\Lambda)} = I(X{:}Y)_{\mathcal{I}\otimes\mathcal{M}'(\Lambda)} \leq I(X{:}YK)_{\mathcal{I}\otimes\mathcal{M}'(\Lambda)} \leq I_{acc}$$

$\Lambda = \sum_x p_x |x\rangle\langle x| \otimes \rho_x$

q info processing ineq

equal and $\mathcal{M}'$ also optimal

How to optimize measurement for Iacc?

2. EB Davies, IEEE Trans Info Th, 24, p596, 1978

For any ensemble of states in d dimensions, $\mathcal{E} = \{p_x, \rho_x\}$
optimal measurement has POVM $\mathcal{M} = \{M_y\}_{y=1}^{n}$ with

(a) rank(M_y) = 1 and

(b) $d \leq n \leq d^2$

Proof (b): see e.g., Watrous book, or 1904.10985 Corollary 5.

Based on:

Caratheodory's Theorem:

Let $S \subseteq \mathbb{R}^t$, conv(S) convex hull of S.

Then, any $x \in$ conv(S) is a convex combination of at most

t+1 elements of S.

How to optimize measurement for Iacc?

3. EB Davies, IEEE Trans Info Th, 24, p596, 1978
   Sasaki, Barnett, Jozsa, Osaki, Hirota 9812062
   Decker 0509122

   Informally: many equiprobable ensembles of states with symmetry have optimal measurement with the same symmetry.

How to optimize measurement for Iacc?

3. Ensembles with symmetry

Example 2. Define the ensemble $\mathcal{E}_1$ with

p(0) = p(1) = p(2) = 1/3, $\rho_x = |\psi_x\rangle\langle\psi_x|$, $|\psi_0\rangle = |0\rangle$

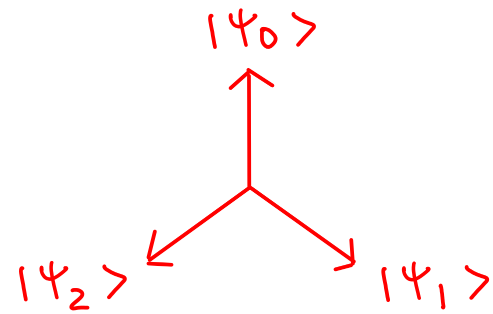
$|\psi_1\rangle = \cos\frac{\pi}{3}|0\rangle + \sin\frac{\pi}{3}|1\rangle$

$|\psi_2\rangle = \cos\frac{\pi}{3}|0\rangle - \sin\frac{\pi}{3}|1\rangle$

9812062: optimal meas has POVM

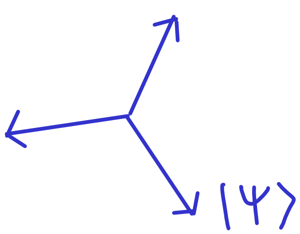
$$\mathcal{M}_1 = \left\{ M_k = \frac{2}{3} R^k |\psi\rangle\langle\psi| R^{k+} \right\}_{k=0,1,2}$$

where $R = e^{i\sigma_y \frac{2}{3}\pi}$ (note $R^k|\psi_0\rangle = |\psi_k\rangle$)



(the trine or "Mercedes" states)

e.g.,



Ex: optimal $|\psi\rangle = |\psi_0^\perp\rangle = |1\rangle$.

So, $M_0 = |\psi_0^\perp\rangle\langle\psi_0^\perp| = |1\rangle\langle 1|$

$M_1 = |\psi_1^\perp\rangle\langle\psi_1^\perp|$, $|\psi_1^\perp\rangle = \sin\frac{\pi}{3}|0\rangle - \cos\frac{\pi}{3}|1\rangle$

$M_2 = |\psi_2^\perp\rangle\langle\psi_2^\perp|$, $|\psi_2^\perp\rangle = \sin\frac{\pi}{3}|0\rangle + \cos\frac{\pi}{3}|1\rangle$

How to optimize measurement for Iacc?

3. Ensembles with symmetry

Example 2. Define the ensemble $\mathcal{E}_1$ with

$p(0) = p(1) = p(2) = 1/3$, $\rho_x = |\psi_x\rangle\langle\psi_x|$, $|\psi_0\rangle = |0\rangle$

$|\psi_1\rangle = \cos\frac{\pi}{3}|0\rangle + \sin\frac{\pi}{3}|1\rangle$

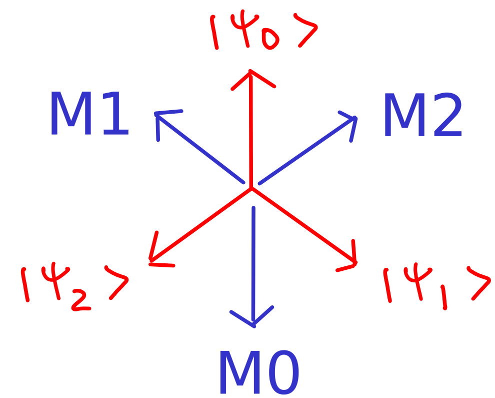
$|\psi_2\rangle = \cos\frac{\pi}{3}|0\rangle - \sin\frac{\pi}{3}|1\rangle$

$\mathcal{M}_1 = \{M_k\}_{k=0,1,2}$

$M_0 = |\psi_0^\perp\rangle\langle\psi_0^\perp| = |1\rangle\langle 1|$

$M_1 = |\psi_1^\perp\rangle\langle\psi_1^\perp|$, $|\psi_1^\perp\rangle = \sin\frac{\pi}{3}|0\rangle - \cos\frac{\pi}{3}|1\rangle$

$M_2 = |\psi_2^\perp\rangle\langle\psi_2^\perp|$, $|\psi_2^\perp\rangle = \sin\frac{\pi}{3}|0\rangle + \cos\frac{\pi}{3}|1\rangle$



Ex: find pr(y|x) for all x,y.

If y=0, pr(x=0|y=0) = 0

pr(x=1|y=0) = pr(x=2|y=0) = 1/2, so H(X|y=0) = 1.

$$H(X|Y) = \underbrace{p(y=0)}_{\frac{1}{3}} \cdot \underbrace{H(X|y=0)}_{1} + \underbrace{p(y=1)}_{\frac{1}{3}} \cdot \underbrace{H(X|y=1)}_{1} + \underbrace{p(y=2)}_{\frac{1}{3}} \cdot \underbrace{H(X|y=2)}_{1} = 1$$

Iacc = H(X) - H(X|Y) = (log 3) - 1 = 0.5850.