QECC condition for stabilizer codes and Pauli errors

Let G1, G2, ..., Gm be the generators for the stabilizer group for a stabilizer code C.

Let E1, E2, ..., Er be a set of Pauli matrices.

Then, any quantum operation with Kraus operators in the span of E1, E2, ..., Er is correctible on C if

$\forall\, i \neq j$, $\exists\, l$ s.t. $E_i E_j$ anticommutes with G_l.

or Ei Ej is in S.

Next: showing the above abstract condition implies that the errors have distinct +/- signs when we measure G1, G2, ..., Gm, leading to a simple algorithm to identify the error,

or errors with identical syndromes act identically on the code space and there is no need to distinguish them.

Claim: measuring G1,G2,...,Gm effects a refinement of the measurement with projectors $E_i P E_i$.

Proof: the measurement of G1, G2, ..., Gm is described by 2^m projectors $\left(\frac{I \pm G_1}{2}\right)\left(\frac{I \pm G_2}{2}\right)\left(\frac{I \pm G_3}{2}\right) \cdots \left(\frac{I \pm G_m}{2}\right)$.

$\forall i$ let $c_{il} = \begin{cases} +1 & \text{if } [E_i, G_l] = 0 \\ -1 & \text{if } \{E_i, G_l\} = 0 \end{cases}$

$$E_i P E_i = E_i \left(\frac{I+G_1}{2}\right)\left(\frac{I+G_2}{2}\right)\left(\frac{I+G_3}{2}\right) \cdots \left(\frac{I+G_m}{2}\right) E_i$$

direct substitution

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$\forall i$ let $c_{i\ell} = \begin{cases} +1 & \text{if } [E_i, G_\ell] = 0 \\ -1 & \text{if } \{E_i, G_\ell\} = 0 \end{cases}$

$$E_i P E_i = E_i \left(\frac{I+G_1}{2}\right)\left(\frac{I+G_2}{2}\right)\left(\frac{I+G_3}{2}\right)\cdots\left(\frac{I+G_m}{2}\right) E_i$$

$$= E_i \left(\frac{I+G_1}{2}\right) E_i E_i \left(\frac{I+G_2}{2}\right) E_i E_i \left(\frac{I+G_3}{2}\right)\cdots\left(\frac{I+G_m}{2}\right) E_i$$

insert identity

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Proof: the measurement of G1, G2, ..., Gm is described by 2^m projectors $\left(\frac{I \pm G_1}{2}\right)\left(\frac{I \pm G_2}{2}\right)\left(\frac{I \pm G_3}{2}\right) \cdots \left(\frac{I \pm G_m}{2}\right)$.

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$$\begin{aligned} E_i P E_i &= E_i \left(\frac{I+G_1}{2}\right)\left(\frac{I+G_2}{2}\right)\left(\frac{I+G_3}{2}\right) \cdots \left(\frac{I+G_m}{2}\right) E_i \\ &= E_i \left(\frac{I+G_1}{2}\right) E_i E_i \left(\frac{I+G_2}{2}\right) E_i E_i \left(\frac{I+G_3}{2}\right) \cdots \left(\frac{I+G_m}{2}\right) E_i \\ &= \left(\frac{I+c_{i1}G_1}{2}\right)\left(\frac{I+c_{i2}G_2}{2}\right)\left(\frac{I+c_{i3}G_3}{2}\right) \cdots \left(\frac{I+c_{im}G_m}{2}\right) \end{aligned}$$

which is one of the projectors when measuring G1,G2,...,Gm.

Note $E_i P E_i$ projects onto c_{i1} eigenspace of G_1, and c_{i2} eigenspace of G_2, ..., c_{im} eigenspace of G_m.

The list $c_{i1}, c_{i2}, \ldots, c_{im}$ is the syndrome (list of outcomes when measuring G1, ...,Gm) if Ei happens.

Furthermore, for $i \neq j$,

if $E_i E_j$ anticommute with some G_ℓ

then, exactly one of Ei, Ej commute with G_ℓ,

and one of Ei, Ej anticommute with G_ℓ.

So, $c_{i\ell} \neq c_{j\ell}$ and Ei, Ej must have different syndromes.

Thus the QECC condition $\forall\, i \neq j, \exists\, \ell$ s.t. $\{E_i E_j, G_\ell\} = 0$

implies that measuring the stabilizer generators measures the syndrome (a simple algorithm).

If Ei Ej is in S, P Ei P = P Ej P but no need to distinguish Ei from Ej.