Lecture VI

The threshold theorem:

Proof and assumptions

Recall : level reduction

Given a circuit C we can construct a FT circuit, which when subjected to local stochastic noise w/ error rate p, is equivalent to C subjected to local stochastic noise w/ error rate $p' = \binom{A}{t+1} p^{t+1}$

locations in largest exRec

If $\binom{A}{t+1} p^{t+1} < p$ then our FT circuit is more reliable than C.

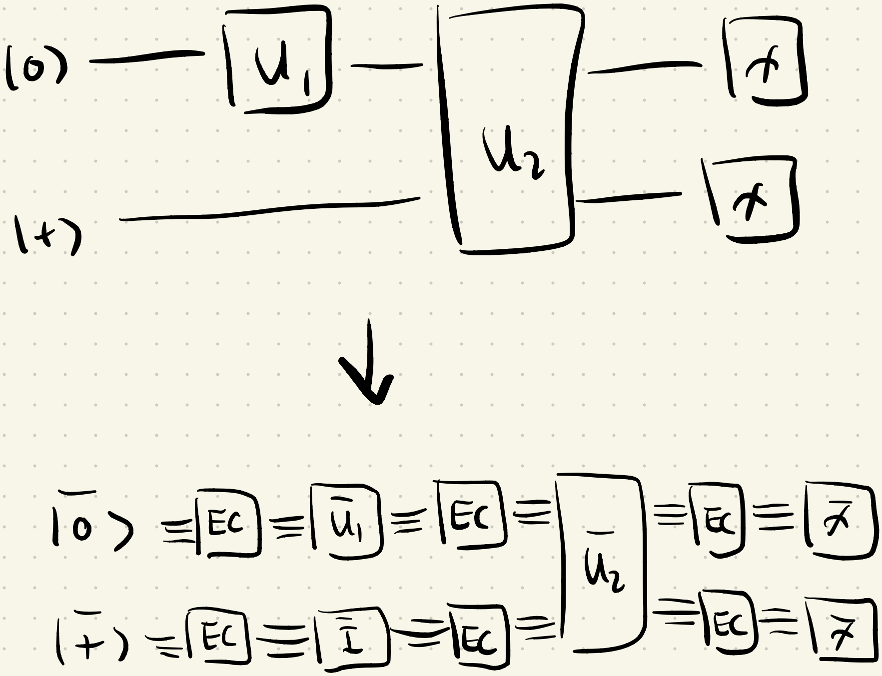
We can repeat this process to further reduce the error rate.

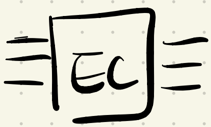
Def: Code concatenation

Given an $[[n, 1, d]]$ code we take each physical qubit of the code and encode it again using the same code, giving an $[[n^2, 1, d^2]]$ code.

Repeating this L times gives a $[[n^L, 1, d^L]]$ code.

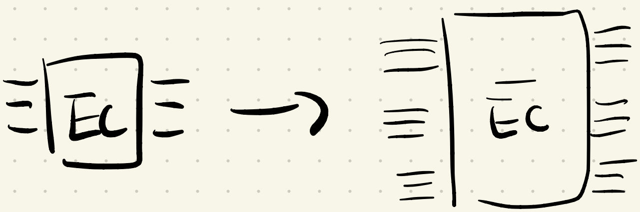
In a similar way we can define a concatenated FT simulation of a circuit.





is a circuit

We encode this again in the same code



Thm: $\exists\, p_T$ such that if a system is subjected to local stochastic noise w/ error prob. $p < p_T$, then for any $\varepsilon > 0$ & any circuit C with T locations, there exist a FT circuit with output distribution within statistical distance ε of the output distribution of C (executed perfectly).

The FT protocol uses resources (time, qubits, gates) that are a factor polylog (T/ε) greater than those of C.

Proof: Idea is to use a concatenated FT sim. w/ L levels.

1st level of concatenation

$$p^{(1)} \le \binom{A}{t+1} p^{t+1} \qquad t = \left\lfloor \frac{d-1}{2} \right\rfloor$$

Define $p_T = 1 \Big/ \binom{A}{t+1}^{1/t}$

$$p^{(1)} \leq p_T \left(\frac{p}{p_T} \right)^{t+1}$$

$$\frac{p^{(1)}}{p_T} = \left(\frac{p}{p_T} \right)^{t+1}$$

2nd level of concatenation

$$\frac{p^{(2)}}{p_T} \leq \left(\frac{p^{(1)}}{p_T} \right)^{t+1}$$

$$= \left(\left(\frac{p}{p_T} \right)^{t+1} \frac{p_T}{p_T} \right)^{t+1}$$

$$= \left(\frac{p}{p_T} \right)^{(t+1)^2}$$

L'th level of concatenation

$$\frac{p^{(L)}}{p_T} \leq \left(\frac{p}{p_T}\right)^{(t+1)^L}$$

If $p < p_T$ we can make the error arbitrarily small by choosing L large enough

We choose

$$L = \lceil \log_{t+1} \log_{p/p_T} (\varepsilon / p_T T) \rceil$$

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$$= \left\lceil \frac{\log_2 \left(\frac{\log_2 (T p_T / \varepsilon)}{\log (p_T / p)} \right)}{\log_2 (t+1)} \right\rceil$$

$$= O(\log \log (T/\varepsilon))$$

This gives

$$p^{(L)} \leq \varepsilon / T$$

$\Rightarrow$ Prob of having a single logical fault is $\leq \varepsilon$ (T locations in circuit) $\square$

This gives a lower bound on the error threshold p_T, but what is p_T in practice? (Billion dollar question!)

Example: Concatenated

Steane code

$d=3 \quad t=7$

$p_T = 1/\binom{A}{2}$

One can calulate e.g.

$A = 679$

$\Rightarrow \binom{A}{2} = 230,181$

$p_T = 4.3 \times 10^{-6}$

Highest proven threshold value is for Knill's scheme where $p_T > 10^{-3}$

In practice people often estimate the threshold using simulations.

(possible due to Gottesman Knill theorem)

For Knill's scheme

$p_T \sim 3\%$

For surface code

$p_T \sim 1\%$

In practice the polylog overhead can hide large constant factors.

e.g. surface code

$\sim 10^3$ physical qubits needed per logical qubit!

But using certain special codes (low-density parity-check codes w/ additional properties) one can show that

FT q. comp. is possible w/ constant overhead!

Reducing the overhead for practical FT schemes is a v. important research problem!

Assumptions behind the threshold thm

1. Same error rates for all locations

Not necessary

We can repeat our proof but now p_T not a number but a surface.

2 Local error model

Necessary

Small-scale correlation is included in local stochastic error model

But long range correlation will kill the threshold

This is a real problem

e.g. cosmic rays in superconducting circuits

3 Long range gates

<u>Not necessary</u>

In concatenated codes we need we naively need long range connections between qubits.

We can avoid this by using SWAP gates or topological codes

(4) Stochastic errors

Not necessary (not fully proven)

There exists a threshold

than for coherent errors

but with a reduced

threshold (sim. for

non - Markovian errors)

But it's not clear if

this is a real effect

or an artefact of the

proof technique.

Coherent errors &
non-Markovian errors
are difficult to simulate,
so we don't have much
numerical evidence one way
or another.