① Registers and states Sec 3.1.1

A register: a component in a computer
a physical system that holds some data (state)

Assume: ① Every register has a unique name and can be distinguished
② " " " a finite nonempty set of classical states

Notation: sans serif font $\mathsf{X}, \mathsf{Y}, \mathsf{Z}, \mathsf{X}_1, \dots, \mathsf{X}_n, \mathsf{Y}_A, \mathsf{Y}_B$ etc.

Let register X have classical state set Σ.

ⓐ We associate the complex Euclidean space $\mathcal{X} = \mathbb{C}^{\Sigma}$ with X.

scripted font, same letter

ⓑ For distinct registers $\mathsf{X}_1, \mathsf{X}_2, \dots, \mathsf{X}_n$, the n-tuple $(\mathsf{X}_1, \mathsf{X}_2, \dots, \mathsf{X}_n) = \mathsf{Y}$ is a register, with classical states $\Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_n$ (cartesian product).

The CES associated with Y: $\mathcal{Y} = \mathbb{C}^{\Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_n} = \mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \dots \otimes \mathcal{X}_n$.

ⓒ State set for register X: $\underset{\text{density operators}}{\mathrm{D}}(\underset{\text{on } \mathcal{X}}{\mathcal{X}}) = \{ \underbrace{\rho \in \mathrm{Pos}(\mathcal{X})}_{\text{universe where elements come from}} : \underbrace{\mathrm{Tr}\, \rho = 1}_{\text{further constraints}} \}$.

- Green: this and next lectures. (the running to warm up before the soccer game)
- "Sec" refers to LN 2011 unless otherwise stated.

② Complex Euclidean space (CES)

②(a) From Rudin (Sec 3.16 3rd Ed)

Let $k \in \mathbb{N}$ (positive integers)

$x_i \in \mathbb{C}$ (or any field), $i = 1 \ldots k$

<u>Def</u>: $x = (x_1, \ldots, x_k)$ (ordered k-tuple)

x: vector; $x_1, \ldots, x_k$: coordinates

$y = (y_1 \ldots, y_k)$

<u>Def</u>: (i) addition $x+y := (x_1+y_1, \ldots, x_k+y_k)$ binary op on x, y, resulting in new vector "$x+y$", whose coordinates are given by RHS

(ii) scalar multiplication $\alpha x := (\alpha x_1, \ldots, \alpha x_k)$

$\alpha \in \mathbb{C}$ "scalar"

Obs:/ <u>Def</u>: set of such vectors with addition & scalar mult. form a <u>vector space</u>.

<u>Def</u>: Inner product $x \cdot y = \sum_{i=1}^{k} \overline{x_i}\, y_i$ (complex conjugate)

<u>Def</u>: Norm of x: $|x| = (x \cdot x)^{\frac{1}{2}} = \left(\sum_{i=1}^{k} |x_i|^2 \right)^{\frac{1}{2}}$ $\boxed{FN}$

Obs:/ <u>Def</u>: Vector space with inner product: inner product space

Normed inner product space: Euclidean space.

Dim = k.

FN: $\|\cdot\|$ is norm if $\forall x, y \in \mathbb{C}^k$, $\alpha \in \mathbb{C}$:

① $\|x\| \geq 0$

② $\|x\| = 0 \iff x = 0$ (additive identity)

③ $\|\alpha x\| = |\alpha| \, \|x\|$

④ $\|x+y\| \leq \|x\| + \|y\|$

②ⓑ From 2011 Lecture notes (LN) Sec 1.1

Def: let Σ be a finite, non-empty set (eg $\Sigma = \{\uparrow, \downarrow\}$)

$\mathbb{C}^\Sigma$ = set of all functions from Σ to $\mathbb{C}$

eg. let $u \in \mathbb{C}^\Sigma$

for $a \in \Sigma$, $u(a)$ = u evaluated on a (def of u as fcn from Σ to $\mathbb{C}$)

= "a-th entry" of u (interpret u as a vector)

eg. $\Sigma = \{1, 2, \dots, k\}$ reduces to Rudin. $\leftarrow \mathbb{C}^k$

$x(i) = x_i$ for $i \in \Sigma$.

Quantifiers

$\forall a \in \Sigma$

Def: $u, v \in \mathbb{C}^\Sigma$, $\alpha \in \mathbb{C}$

(i) Addition $(u+v) \in \mathbb{C}^\Sigma$ defined by $(u+v)(a) = u(a) + v(a)$

(ii) scalar mult: $\alpha u \in \mathbb{C}^\Sigma$ " " $(\alpha u)(a) = \alpha \cdot u(a)$.

Def: Inner product of $u, v \in \mathbb{C}^\Sigma$:

$$\langle u, v \rangle = \sum_{a \in \Sigma} \overline{u(a)} \cdot v(a)$$

NB. $\mathbb{C}^\Sigma$ is a CES that generalizes $\mathbb{C}^{|\Sigma|}$. Dim $(\mathbb{C}^\Sigma) = |\Sigma|$.

Def: Euclidean norm of $u \in \mathbb{C}^\Sigma$

$$\| u \| = \sqrt{\langle u, u \rangle} = \sqrt{\sum_{a \in \Sigma} |u(a)|^2}$$

Ex: read Sec 1.1.2 for properties of the Euclidean norm.

Def: For $1 \leq p < \infty$, p-norm of u:

$$\| u \|_p = \left(\sum_{a \in \Sigma} |u(a)|^p \right)^{\frac{1}{p}}$$

($p=2$ gives EN).

Def: $\| u \|_\infty = \max \{ |u(a)| : a \in \Sigma \}$

(? reading)

Thm: Cauchy-Schwarz inequality

$\forall u, v \in \mathbb{C}^{\Sigma}, \quad |\langle u, v \rangle| \le \|u\| \, \|v\|$

"=" iff u, v lin dependent.

Thm: Hölder's ineq

$\forall u, v \in \mathbb{C}^{\Sigma}, \quad p, q \in [1, \infty]$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$

$|\langle u, v \rangle| \le \|u\|_p \, \|v\|_q$

Def: The set $\{ u_l : \underset{\text{index}}{l} \in \underset{\substack{\text{index set} \\ \text{non empty}}}{\Gamma} \} \subseteq \mathbb{C}^{\Sigma}$

- is orthogonal if $l \neq m \Rightarrow \langle u_l, u_m \rangle = 0$
- is orthonormal if $\langle u_l, u_m \rangle = \delta_{l,m}$ (Kronecker-delta)

Notation: standard basis for $\mathbb{C}^{\Sigma}$: $\{ e_a : a \in \Sigma \}$

where $\underset{|a\rangle}{e_a}(b) = \begin{cases} 1 & \text{if } a = b. \\ 0 & \text{if } a \neq b. \end{cases}$

"only a-th coordinate is 1, other coordinates 0".

2 c Combining CESs (LN Sec 1.2.3 – 1.2.4)

Def Let $\mathcal{X}_i = \mathbb{C}^{\Sigma_i}$, for $i = 1, \dots, n$, be n CESs.

Their <u>direct sum</u>, denoted $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \dots \oplus \mathcal{X}_n$ is $\mathbb{C}^{\Delta}$

where $\Delta = \bigcup_{i=1}^{n} \{ (i, a_i) : a_i \in \Sigma_i \}$.

eg. $n = 2$, $\Sigma_1 = \{\uparrow, \downarrow\}$, $\Sigma_2 = \{0, 1, 2\}$

$$\left.\begin{array}{l} \\ \Delta = \{ (1,\uparrow), (1,\downarrow), (2,0), (2,1), (2,2) \} \end{array}\right\} \mathbb{C}^{\Delta} \cong \mathbb{C}^5$$

eg. Let $u_i \in \mathbb{C}^{\Sigma_i}$, $i = 1, \dots, n$

$u = u_1 \oplus u_2 \oplus \dots \oplus u_n$ is the vector s.t.

$\forall j \in 1, \dots, n$, $\forall a_j \in \Sigma_j$, $u(j, a_j) = u_j(a_j)$.

ie. $u = \begin{bmatrix} \boxed{u_1} \\ \boxed{u_2} \\ \vdots \\ \boxed{u_n} \end{bmatrix}$

$\mathrm{Dim}(\mathcal{X}_1 \oplus \dots \oplus \mathcal{X}_n) = |\Sigma_1| + \dots + |\Sigma_n| = \dim(\mathcal{X}_1) + \dots + \dim(\mathcal{X}_n)$.

Ex: Prove : ① $(u_1 \oplus u_2 \oplus \dots \oplus u_n) + (v_1 \oplus v_2 \oplus \dots \oplus v_n) = (u_1 + v_1) \oplus (u_2 + v_2) \oplus \dots \oplus (u_n + v_n)$

② $\alpha (u_1 \oplus u_2 \oplus \dots \oplus u_n) = (\alpha u_1) \oplus (\alpha u_2) \oplus \dots \oplus (\alpha u_n)$

③ $\langle u_1 \oplus \dots \oplus u_n, v_1 \oplus \dots \oplus v_n \rangle = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle + \dots + \langle u_n, v_n \rangle$

Qn: What constitute a proof?

How to quantify the above? (eg in ②, $\forall \alpha \in \mathbb{C}$, $\forall u_i \in \mathcal{X}_i$)

(*)

Def Let $\mathcal{X}_i = \mathbb{C}^{\Sigma_i}$ for $i=1,\dots,n$ be n CESs.

Their <u>tensor product</u>, denoted $\underset{\substack{\uparrow \\ \text{binary op on 2 CESs}}}{\mathcal{X}_1 \otimes} \mathcal{X}_2 \otimes \cdots \otimes \mathcal{X}_n$, is $\mathbb{C}^{\overbrace{\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n}}$.

defined in (2)(1)

eg1 $\Sigma_1 = \{\uparrow, \downarrow\}$, $\Sigma_2 = \{0,1,2\}$

$u \in \mathbb{C}^{\Sigma_1 \times \Sigma_2}$ is a function from $\{\uparrow 0, \uparrow 1, \uparrow 2, \downarrow 0, \downarrow 1, \downarrow 2\}$ to $\mathbb{C}$

eg. $u(\uparrow 0) = \frac{1}{\sqrt{2}} = u(\downarrow 2)$, $u(\uparrow 1) = u(\uparrow 2) = u(\downarrow 0) = u(\downarrow 1) = 0$

In bra-ket notation, $|u\rangle = \left(|\uparrow 0\rangle + |\downarrow 2\rangle\right)\frac{1}{\sqrt{2}}$.

(*)(*)

Def: If $u_i \in \mathcal{X}_i$ for $i=1,\dots,n$, the symbol $u_1 \otimes u_2 \otimes \cdots \otimes u_n$ denotes the vector whose $(a_1, a_2, \dots, a_n)$-entry is $u_1(a_1)\, u_2(a_2) \cdots u_n(a_n)$.

Obs: Could instead define $\mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \cdots \otimes \mathcal{X}_n$ to be the CES spanned by $u_1 \otimes u_2 \otimes \cdots \otimes u_n$.

rep as column vector, basis $\begin{matrix}\uparrow \\ \downarrow\end{matrix}$, basis $\begin{matrix}0\\1\\2\end{matrix}$

eg.2 Following eg 1, if $u_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $u_2 = \begin{bmatrix}3\\4\\5\end{bmatrix}$,

then $u_1 \otimes u_2 = \begin{bmatrix} 1\begin{bmatrix}3\\4\\5\end{bmatrix} \\ \hline 2\begin{bmatrix}3\\4\\5\end{bmatrix}\end{bmatrix} \begin{matrix}\text{block } \uparrow \\ \\ \text{block } \downarrow\end{matrix} = \begin{bmatrix}3\\4\\5\\ \hline 6\\8\\10\end{bmatrix}$

1st tensor compont

Note how concise (*) & (*)(*) are!

Ex prove the three identities top of p8.
Carefully explain what is the LHS & the RHS, and what has to be proved.

3 Linear operators (Sec 1.2).

Def: Given CESs $\mathcal{X}, \mathcal{Y}$, $L(\mathcal{X},\mathcal{Y})$ = set of all linear operators from $\mathcal{X}$ to $\mathcal{Y}$.

or mappings, maps, transformations, functions, ...

Notation: for $A \in L(\mathcal{X},\mathcal{Y})$, $u \in \mathcal{X}$, we write Au for $A(u)$.
Reason: matrix interpretation of both A & u.

defines $(A+B)$ on all u, thus defines $A+B$

Obs: $L(\mathcal{X},\mathcal{Y})$ is a vector space with

+ in $\mathcal{Y}$

(i) addition, where $(A+B)u := Au + Bu$, $\forall A, B \in L(\mathcal{X},\mathcal{Y}), u \in \mathcal{X}$

in $\mathcal{Y}$

(ii) Scalar multi, where $(\alpha A)u := \alpha(Au)$

scalar multi in $\mathcal{Y}$.

Proof: need to show if A, B linear, then $A+B$, αA are.

Matrix representation of linear operators:

Def: Let $\Sigma_{in}, \Sigma_{out}$ be non-empty, finite sets.

Let $M_{\Sigma_{out}, \Sigma_{in}}(\mathbb{C})$ denote the set of all complex matrices,

each element is a function $M: \Sigma_{out} \times \Sigma_{in} \to \mathbb{C}$.

detail: P4-5 LN

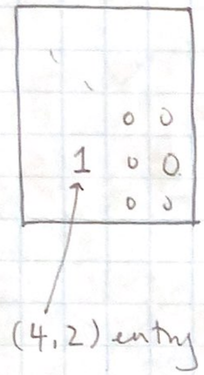
Cor: $M_{\Sigma_{out}, \Sigma_{in}}$ is a CES $\mathbb{C}^{\Sigma_{out} \times \Sigma_{in}}$.

row index   column index

Def: $M(a,b)$ = (a,b)-entry of M, "a-th row", "b-th column".

Qn: Let $M(a,b) = 1$, $M(k,l) = 0$ $\forall (k,l) \neq (a,b)$.

Is $M = |a\rangle\langle b|$ or $|b\rangle\langle a|$?

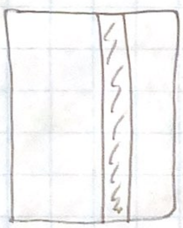
eg. $M =$ 

takes $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ to $\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}$ so $M = |4\rangle\langle 2|$

$|2\rangle$ $\qquad$ $|4\rangle$

output symbol = row index; input symbol = column index

and $|v\rangle$ to 0 if $\langle v|2\rangle = 0$.

eg. $M =$  , the b-th column is $M|b\rangle$

General.

eg. $M = \sum_{a,b} |a\rangle\langle a| M |b\rangle\langle b| = \sum_{a,b} \underbrace{\langle a|M|b\rangle}_{= M(a,b)} |a\rangle\langle b|$.

Def: take standard basis $\{e_a\}$ for $\mathcal{Y}$, $\{e_b\}$ for $\mathcal{X}$. Then the matrix rep for $A \in L(\mathcal{X},\mathcal{Y})$, denoted M_A, is given by

$$\langle e_a, A e_b\rangle = a\text{th entry of } Ae_b = M_A(a,b)$$

NB. $L(\mathcal{X},\mathcal{Y}) \sim M_{\Sigma_{out} \times \Sigma_{in}}(\mathbb{C})$

Notation: $L(\mathcal{X},\mathcal{Y})$ has an o.n. basis $\{E_{a,b}\}_{a \in \Sigma_{out},\, b \in \Sigma_{in}}$

$E_{a,b} = e_a e_b^* = |a\rangle\langle b|$, $\quad E_{a,b}(k,l) = \begin{cases} 1 & \text{if } (a,b) = (k,l) \\ 0 & \text{otherwise.} \end{cases}$

adjoint or dagger

Reading Ex: multiplication rules.

④ On operators and matrices ---- Sec 1.2 – 1.5, 2.1 – 2.3.

(i) $\dim(L(\mathcal{X},\mathcal{Y})) = \dim(\mathcal{X})\dim(\mathcal{Y})$

(ii) def: for $A \in L(\mathcal{X},\mathcal{Y})$

- Kernel $\mathrm{Ker}(A) = \{u \in \mathcal{X} : Au = 0\}$
- Image $\mathrm{Im}(A) = \{Au \in \mathcal{Y} : u \in \mathcal{X}\}$
- Rank $\mathrm{rank}(A) = \dim(\mathrm{Im}(A))$

Thm: $\dim(\mathrm{Ker}(A)) + \mathrm{rank}(A) = \dim(\mathcal{X})$

(iii) def: for $A \in L(\mathcal{X},\mathcal{Y})$ (denote matrix rep as A also)

- Entry-wise conjugate : $\overline{A}$ (s.t. $\overline{A}(a,b) = \overline{A(a,b)}$)
- Transpose : A^T (s.t. $A^T(a,b) = A(b,a)$)
- Adjoint : A^* (s.t. $\langle v, Au\rangle = \langle A^* v, u\rangle$)

$\quad = A^\dagger$ in physics

in $L(\mathcal{Y},\mathcal{X})$

(iv) view vector $u \in \mathcal{X} = \mathbb{C}^\Sigma$ as matrix in $M_{\Sigma,\{1\}}$

$\overline{u}, u^T, u^*$ are defined as in (iii).

Note $u^* v = \langle u, v\rangle$. $u \in L(\mathbb{C},\mathcal{X}) = \mathcal{X}$

$u^* \in L(\mathcal{X},\mathbb{C})$ = dual space of $\mathcal{X} = \mathcal{X}^*$

Reading exercise:

- Sec 1-2.3 on operators from direct sum of CESs to direct sum of CEs.
- Sec 2.2.1 - - - - - - - tensor product - - - - tensor product - - -.

Qn: is this the same as direct sum of operator space ??

(v) Def: $L(\mathcal{X}) := L(\mathcal{X}, \mathcal{X})$.

$\mathbb{1} \in L(\mathcal{X})$ identity operator (ie $\mathbb{1}u = u \ \forall u \in \mathcal{X}$).

or $\mathbb{1}_{\mathcal{X}}$

$A \in L(\mathcal{X})$ invertible if $\exists B \in L(\mathcal{X})$ s.t. $AB = BA = \mathbb{1}$

call A^{-1}

General linear group $GL(\mathcal{X}) = \{A \in L(\mathcal{X}) : A \text{ invertible}\}$.

(vi) Def: $A \in L(\mathbb{C}^{\Sigma})$, trace $Tr(A) := \sum_{a \in \Sigma} A(a,a)$.

determinant $Det(A) := \ldots$

Thm: Tr is a linear fn on $L(\mathbb{C}^{\Sigma})$

Thm: $\forall A \in L(\mathcal{X}, \mathcal{Y})$, $B \in L(\mathcal{Y}, \mathcal{X})$, $tr(BA) = tr(AB)$.

(vii) For CES $L(\mathcal{X}, \mathcal{Y}) \sim M_{\Sigma_y \times \Sigma_x}$ or $\mathbb{C}^{\Sigma_y \times \Sigma_x}$, $A, B \in L(\mathcal{X}, \mathcal{Y})$

the inner product $\langle A, B \rangle = Tr(A^* B)$

$\langle A, B \rangle$ viewed as $\dim(\mathcal{X}) \cdot \dim(\mathcal{Y})$ - dim vector

NB: Trace is defined wrt a basis but it is basis independent.

- Summary for lecture 1:

- Sec 3.1.1, Sec 1.1 – 1.3.1, 2.2.1

- CESs, eg $\mathbb{C}^{\Sigma}$, $L(\mathbb{C}^{\Sigma_{in}}, \mathbb{C}^{\Sigma_{out}}) \sim M_{\Sigma_{out} \times \Sigma_{in}}(\mathbb{C})$

  add, s. multi, inner products, norms, ...