QIC 820 Part 1 lecture 2

Last time:

$$u \in \mathcal{X} \xrightarrow{\quad A \quad} v \in \mathcal{Y}$$

vector

$$\begin{array}{c} A \\ \rotatebox{90}{$\in$} \\ L(\mathcal{X},\mathcal{Y}) \end{array}$$

operators

(as vector of dim = dim($\mathcal{X}$) · dim($\mathcal{Y}$))

Today: Properties of operators and matrix analysis

(Sec1.3.2) ① Eigenvectors and eigenvalues (evecs & evals)

Def: if $A \in L(\mathcal{X})$, $u \in \mathcal{X}$, $\underset{\substack{| \\ \text{zero vector}}}{u \neq 0}$, $Au = \lambda u$ for some $\lambda \in \mathbb{C}$

then u is an eigenvector of A, and λ the corr eigenvalue.

Def: fix $A \in L(\mathcal{X})$. Consider $f(z) = \underset{\substack{/ \\ \text{determinant}}}{\mathrm{Det}}(z \mathbb{1}_{\mathcal{X}} - A)$.

① $f(z)$ is a polynomial in z with deg = dim ($\mathcal{X}$).

② By the unique factorization theorem on $f(z)$

$$f(z) = (z-\lambda_1)(z-\lambda_2) \cdots (z-\lambda_{\dim(\mathcal{X})}) \quad \text{for } \lambda_i \in \mathbb{C}$$

③ $\forall i = 1, 2, \ldots, \dim(\mathcal{X})$, $f(\lambda_i) = \mathrm{Det}(\lambda_i \mathbb{1}_{\mathcal{X}} - A) = 0$

Recall $\mathrm{Det}(M) \neq 0 \iff M$ invertible, so $\lambda_i \mathbb{1}_{\mathcal{X}} - A$ not invertible, and $\exists u_i \in \mathrm{Ker}(\lambda_i \mathbb{1}_{\mathcal{X}} - A)$ so u_i, λ_i are evec, eval of A.

Def: Multiset $\{\lambda_1, \lambda_2, \ldots, \lambda_{\dim(\mathcal{X})}\}$ is called the spectrum of A
\# times λ repeats in the set is called the multiplicity of λ.

Thm: $\mathrm{Tr}(A) = \sum_{\lambda \in \mathrm{spec}(A)} \lambda$

$\mathrm{Det}(A) = \prod_{\lambda \in \mathrm{spec}(A)} \lambda$

(Sec 1.4)② Consider $L(\mathcal{X})$.

positive definite operators	$\subseteq$	positive semidef operators	$\subseteq$	Hermitian operators
$\mathrm{Pd}(\mathcal{X})$		$\mathrm{Pos}(\mathcal{X})$		$\mathrm{Herm}(\mathcal{X})$
$\shortparallel$		$\shortparallel$		$\shortparallel$
$\{A \in \mathrm{Pos}(\mathcal{X}) : \mathrm{Det}(A) \neq 0\}$		$\{B^* B : B \in L(\mathcal{X})\}$		$\{A \in L(\mathcal{X}) : A = A^*\}$
$\mathrm{spec}(A) \subseteq \mathbb{R}^+$		$\mathrm{spec}(B^* B) \subseteq \mathbb{R}^+ \cup \{0\}$		NB $\mathrm{spec}(A) \subseteq \mathbb{R}$ (FN 1)
Ex: read 1.4.4		Can change $B \in L(\mathcal{X})$ to $B \in L(\mathcal{X}, \mathcal{Y})$ for some $\mathcal{Y}$.		Ex: read 1.4.2
		Ex: read 1.4.3		

Notation: $A \geqslant 0$ if $A \in \mathrm{Pos}(\mathcal{X})$

$A \geqslant B$ if $A - B \in \mathrm{Pos}(\mathcal{X})$

(the CES $\mathcal{X}$ is omitted)

eg lec 1, register $\leftrightarrow$ $\mathcal{X} = \mathbb{C}^{\Sigma}$

states $\leftrightarrow$ $\underbrace{D(\mathcal{X})}_{\text{density operators on } \mathcal{X}} = \{\rho \in \mathrm{Pos}(\mathcal{X}) : \mathrm{Tr}\,\rho = 1\}$.

P2LN

FN 1: $\langle u, v\rangle \overset{\textcircled{3}}{=} \overline{\langle v, u\rangle}$, so if $\overset{\textcircled{2}}{A = A^*}$, $\overset{\textcircled{1}}{Au = \lambda u}$, then

$$\lambda \langle u,u\rangle \underset{\textcircled{1}}{=} \langle u, Au\rangle \underset{\text{def of } *}{=} \langle A^* u, u\rangle \underset{\textcircled{2}}{=} \langle Au, u\rangle \underset{\textcircled{3}}{=} \overline{\langle u, Au\rangle} = \overline{\lambda}\, \overline{\langle u,u\rangle}$$

but $\langle u,u\rangle$ positive real !, $\lambda = \overline{\lambda}$. (why we need $u \neq 0$)

<u>Def:</u> A projector (or projection, orthogonal projecton, projection operator) is an operator $P \in \mathrm{Pos}(\mathcal{X})$ satisfying $P^2 = P$ (equivalently $\mathrm{spec}(P) = \{0,1\}$).

<u>Def:</u> For a subspace $\mathcal{V} \subseteq \mathcal{X}$, $\Pi_{\mathcal{V}}$ = projector whose image is $\mathcal{V}$.

<u>Def:</u> $A \in L(\mathcal{X}, \mathcal{Y})$ is an <u>isometry</u> if $\forall u \in \mathcal{X}$, $\|Au\| = \|u\|$.

<u>Thm:</u> $A \in L(\mathcal{X}, \mathcal{Y})$ is an isometry

$\Leftrightarrow A^* A = \mathbb{1}_{\mathcal{X}}$

$\Leftrightarrow \langle Au, Av \rangle = \langle u, v \rangle \quad \forall u, v \in \mathcal{X}$. (Pf: Ex)

<u>Def:</u> $A \in L(\mathcal{X})$ normal if $AA^* = A^*A$.

eg all Hermitian operators are normal.

(Sec1.5) ③ The spectral theorem and functions of normal operators.

<u>Thm</u> (the spectral thm) If $\mathcal{X}$ CES, $A \in L(\mathcal{X})$ normal with <u>distinct</u> eigenvalues $\lambda_1, \dots, \lambda_K$, then $\exists!$ projectors $P_1, \dots, P_K \in \mathrm{Pos}(\mathcal{X})$ s.t.

$\overset{\uparrow}{\text{unique}}$

① $P_i P_j = 0$ if $i \neq j$

② $P_1 + \cdot + P_K = \mathbb{1}_{\mathcal{X}}$

③ $\mathrm{rank}(P_i)$ = multiplicity of λ_i

④ $\sum_{i=1}^{K} \lambda_i P_i = A$

⑤ if $\mathrm{spec}(A) = \{\lambda_1, \lambda_2, \dots, \lambda_K, \dots, \lambda_{\dim(\mathcal{X})}\}$, then $\exists$ o.n. basis $\{x_i\}$ for $\mathcal{X}$ s.t. $\sum_{i=1}^{n} \lambda_i x_i x_i^* = A$

the spectral decompositions, ④ unique, ⑤ not nec.

<u>Thm</u> let $A, B \in L(\mathcal{X})$ be <u>commuting</u> <u>normal</u> operators.

Then $\exists$ o.n. basis $\{x_i\}$ of $\mathcal{X}$ s.t.

$$A = \sum_{i=1}^{n} \lambda_i x_i x_i^*, \qquad B = \sum_{i=1}^{n} \mu_i x_i x_i^*$$

are spec decomps of A, B respectively.

<u>Def</u>: If $A \in L(X)$ normal with spec decomp $A = \sum_{i=1}^{K} \lambda_i P_i$,

then a function $f: \mathbb{C} \to \mathbb{C}$ can be extended to A as $f(A) = \sum_{i=1}^{K} f(\lambda_i) P_i$.

eg. If $A \in Pd(X)$, then $\log(A) = \sum_{i=1}^{K} \log(\lambda_i) \; P_i$.

eg. if $A \in Pos(x)$, then $A^{\frac{1}{2}} = \sum_{i=1}^{K} \lambda_i^{\frac{1}{2}} P_i$.

NB: $\sum_{i=1}^{n} f(\lambda_i) \, x_i x_i^* = \sum_{i=1}^{K} f(\lambda_i) P_i$, ie ④⑤ give the same $f(A)$

Qn: what about $A \in L(X, Y)$, $\dim(X) \neq \dim(Y)$?

Sec2.1 <u>Thm</u> (singular-value thm): $A \in L(X, Y)$, $A \neq 0$, $r = \text{rank}(A) \geq 1$.

$\exists$ $s_1, \ldots, s_r \in \mathbb{R}^+$, o.n sets $\{x_1, \ldots, x_r\} \subseteq \mathcal{X}$, $\{y_1, \ldots, y_r\} \subseteq \mathcal{Y}$

s.t. $$\underbrace{A = \sum_{i=1}^{r} \underbrace{s_i}_{\text{singular values.}} y_i x_i^*}_{\text{Singular value decomposition SVD}} \qquad \left(\sum_{i=1}^{r} s_i \underbrace{|y_i\rangle}_{\text{left -}} \underbrace{\langle x_i|}_{\text{right - singular vectors.}} \right)$$

Obs For $A \in L(\mathcal{X},\mathcal{Y})$, $A = \sum_{i=1}^{r} s_i y_i x_i^*$ SVD

$AA^* = \sum_{i=1}^{r} s_i^2 y_i y_i^*$ and $A^*A = \sum_{i=1}^{r} s_i^2 x_i x_i^*$ are spec decomps.

Recipe for SVD: ① find AA^* and its spectral decomp $\sum_{i=1}^{r} \lambda_i y_i y_i^*$

② find x_i's st. $A = \sum_{i=1}^{r} \sqrt{\lambda_i}\, y_i x_i^*$

or find A^*A & spec decomp, and find y_i's.

<u>Cor of SVD</u>: if $A \in L(\mathcal{X},\mathcal{Y})$; $A = \sum_{i=1}^{r} s_i y_i x_i^*$ SVD

then $B = \sum_{i=1}^{r} s_i^{-1} x_i y_i^*$ satisfies $BA = \mathbb{1}_{\mathrm{Im}(A)}$

Moore-Penrose pseudo inverse of A.
Sec 2.1.1.

(Sec 2.3) ④ Schatten norms of operators

<u>Def</u>: $\forall A \in L(\mathcal{X},\mathcal{Y})$, $p \in \mathbb{R}$, $p \geq 1$,

function on normal operator

Schatten p-norm of A : $\|A\|_p := \left(\mathrm{Tr}\left[\overbrace{(A^*A)}^{\text{normal op}}{}^{\frac{p}{2}} \right] \right)^{\frac{1}{p}}$

$\|A\|_\infty := \max\{\|Au\| : u \in \mathcal{X}, \|u\| = 1\}$

Euclidean norm

<u>Obs</u>: if $s(A)$ = vector of singular values of A

then $\|A\|_p = \underbrace{\|s(A)\|_p}_{\text{p-norm on vector}}$

<u>Obs</u>: $\|A\|_2 = \mathrm{Tr}(A^*A)^{\frac{1}{2}} = \left(\sum_{a,b} |A(a,b)|^2 \right)^{\frac{1}{2}}$ = Euclidean norm in $L(\mathcal{X},\mathcal{Y})$.

Qn: Is $\|A\|_p$ the p-norm for elents of $L(\mathcal{X},\mathcal{Y})$ for $p \neq 2$?

Special Schatten p-norms:

$\|\cdot\|_{tr} = \|\cdot\|_1$ trace norm

$\|\cdot\|_F = \|\cdot\|_2$ Frobenius norm

$\|\cdot\| = \|\cdot\|_\infty$ Spectral norm / operator norm

Thm: $\forall A \in L(\mathcal{X})$, $\|A\|_1 = \max\{ \underbrace{|\langle A, U\rangle|}_{\text{objective function}} : \overbrace{U \in \mathrm{U}(\mathcal{X})}^{\text{constraint}} \}$

unitary operator on $\mathcal{X}$ (isometric ops from $\mathcal{X}$ to $\mathcal{X}$)

Other expressions for the optimization $\longrightarrow$

$$\max \ |\langle A, U\rangle| \quad \text{subj to} \ U \in \mathrm{U}(\mathcal{X})$$

$$\max_{U \in \mathrm{U}(\mathcal{X})} |\langle A, U\rangle|$$

Pf: If $A = 0$, $\|A\|_1 = (\mathrm{Tr}\, 0)^1 = 0$, RHS $= 0$ ∴ Claim is true.

If $A \neq 0$, use SVT and obtain SVD for A:

$$A = \sum_{i=1}^{r} s_i y_i x_i^*$$ where $r = \mathrm{rank}(A)$, $\{x_i\}_{i=1}^r$, $\{y_i\}_{i=1}^r$ o.n sets in $\mathcal{X}$

- $\|A\|_1 = \sum_{i=1}^{r} s_i$
- On the RHS, take $U = \sum_{i=1}^{r} y_i x_i^* + \Pi_{\mathrm{Ker}(A)}$

  then $\langle A, U\rangle = \sum_{i=1}^{r} s_i$ ∴ LHS $\leq$ RHS.

- meanwhile, apply spectal decomp to $U \in U(X)$ (normal);

$$U = \sum_{j=1}^{\dim(X)} w_j w_j^* e^{i\theta_j}, \quad \{w_j\}_{j=1}^{\dim(X)} \text{ o.n. basis of } X$$

$$\begin{aligned}\langle A, U\rangle &= tr\left(A^* U\right)\\ &= tr \sum_{j=1}^{\dim X}\sum_{i=1}^{r} s_i\, x_i y_i^* \cdot w_j w_j^*\; e^{i\theta_j}\\ &= \sum_{i=1}^{r} s_i \sum_{j=1}^{\dim(X)} \underbrace{\left(w_j^* x_i\right)}_{\substack{\text{coeff of}\\ x_i \text{ in the}\\ \text{basis}\{w_j\}}}\underbrace{\left(y_i^* w_j\right)}_{\substack{\text{conjugate of}\\ \text{coeff of } y_i \text{ in}\\ \text{the basis}\{w_j\}}} e^{i\theta_j}\end{aligned}$$

$$\underbrace{\underbrace{e^{i\theta_j}\,\langle w_j | x_i\rangle}_{\substack{\text{still the coeff}\\ \text{of a unit vector}\\ \text{in the } \{w_j\} \text{ basis.}}} \cdot \overline{\langle w_j | y_i\rangle}}_{\substack{\text{inner product between 2 unit vectors}\\ \text{absolute value} \le 1}}$$

$$\therefore\ |\langle A, U\rangle| \underset{\substack{\uparrow\\ \text{triangle ineq}\\ \text{over the sum over } i}}{\le} \sum_{i=1}^{r} s_i \left| \qquad \right| \le \sum_{i=1}^{r} s_i = LHS.$$

□

NB We will use the thm to show monotonicity of fidelity later, after defining partial tracing.

eg. $\forall\, u, v \in \mathcal{X}$, show that $\| uu^* - vv^* \|_p = 2^{\frac{1}{p}} \sqrt{1 - |\langle u, v \rangle|^2}$

Pf Let $A = uu^* - vv^*$ $\quad (|u\rangle\langle u| - |v\rangle\langle v|)$

$A \in \mathrm{Herm}(\mathcal{X})$

$\mathrm{tr}\, A = 0$, $\mathrm{rank}(A) \leq 2$

$\mathrm{spec}(A) = \{ \lambda, -\lambda, 0, 0 - - . . 0 \}$

$\mathrm{tr}\, A^2 = \mathrm{tr}\, (|u\rangle\langle u| - |v\rangle\langle v|)(|u\rangle\langle u| - |v\rangle\langle v|)$

$= 2 - 2 |\langle u | v \rangle|^2$

$= 2 \lambda^2$

$\therefore \lambda = \sqrt{1 - |\langle u, v \rangle|^2}$

$\therefore \| A \|_p = (\lambda^p + \lambda^p + 0^p + .. + 0^p)^{\frac{1}{p}} = 2^{\frac{1}{p}} \lambda$

$= 2^{\frac{1}{p}} \sqrt{1 - |\langle u, v \rangle|^2}$.

Ex: Read other properties of $\| \cdot \|_p$ in Sec 2.3.1.

So far covered :

Sec 1.3.2, 1.4, 1.5, 2.1, 2.3

Postpone :

Sec 2.2, 2.4 to later

Rest of lecture 2 : Highlights of Sec 2.5
Reading Ex on what we cannot cover.

⑤ $\|\cdot\|$ any norm, take Euclidean norm for concreteness

<u>Def:</u> X, Y CESs, $A \subseteq X$, $f : A \to Y$ function

f is <u>continuous</u> (cts) at u

if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\left[\forall v \in A \text{ s.t. } \|v - u\| < \delta \right]$

$\|f(v) - f(u)\| < \varepsilon$.

ie if v is $\underbrace{\text{close enough}}_{\text{no further than } \delta}$ to u, then $\underbrace{f(v) \text{ is close to } f(u)}_{\text{no further than } \varepsilon}$

Here δ depends on ε.

set of natural numbers

<u>Def:</u> A set $A \subseteq X$ is compact if every sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq A$

has a subsequence $\{x_{n_i}\}_{i \in \mathbb{N}}$ that convergences to a point $v \in A$.

$(n_1 \leq n_2 \leq \dots\dots)$

<u>Thm:</u> (Heine-Borel Thm) X CES.

complement of A in X

$A \subseteq X$ is compact $\Leftrightarrow$ ① A is closed (ie $X \setminus A$ is open.)

② A is bound (ie $\exists r$ s.t. $\forall x \in A$, $\|x\| < r$)

<u>Why continuity and compactness</u>?

They prevent a lot of "pathologies" that prevents your steps in a proof to hold.

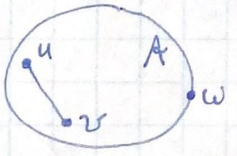
eg If X CES, $A \subseteq X$ compact, then

① if $f: A \to \mathbb{R}$ is continuous on A then f attains a max and a min on A (rather than sup / inf).

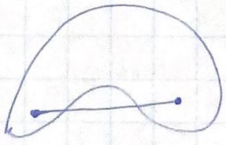
② if $f: X \to Y$ is continuous on A, Y CES, then $f(A) \subseteq Y$ is compact.

$$f(A) = \{f(x) \in Y : x \in A\}$$

<u>Def</u>: X CES, $A \subseteq X$ is <u>convex</u> if $\forall u, v \in A$, $\lambda \in [0,1]$
$\lambda u + (1-\lambda) v \in A$



convex



not convex

<u>Def</u>: for a convex set A, $w \in A$ is an <u>extreme point of A</u>

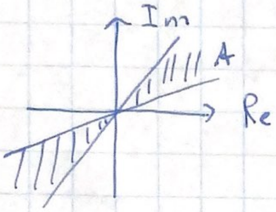
if "$w = \lambda u + (1-\lambda) v$ for some $u, v \in A$" $\Rightarrow$ $u = v = w$.

<u>Def</u>: $A \subseteq X$ is a <u>cone</u> if $\forall u \in A$, $\lambda \geq 0$, $\lambda u \in A$.

Qn cone $\Rightarrow$ convex?

Def $A \subseteq \mathcal{X}$ is a cone if $\forall u \in A,\ \lambda \geq 0,\ \lambda u \in A$.

eg. $\mathcal{X} = \mathbb{C}$



NB cone $\not\Rightarrow$ convex.

So we say "convex cone" if the cone is convex.

Ex: Show that if $A, B \subseteq \mathcal{X}$ convex, then

$$\underline{A}+B = \{u \underline{+} v : u \in A,\ v \in B\}$$ are also convex.

eg. $\mathcal{X}$ CES, $\mathrm{Pos}(\mathcal{X})$ convex cone with only one extreme point 0.

$\mathrm{D}(\mathcal{X})$ convex but not a cone, with rank 1 (pure states) being extreme points.

Def: Σ finite non-empty set.

$p \in \mathbb{R}^{\Sigma}$ prob vector if $p(a) \geq 0$, $\sum_{a \in \Sigma} p(a) = 1$.

Def: A convex combination of points in A is a finite sum

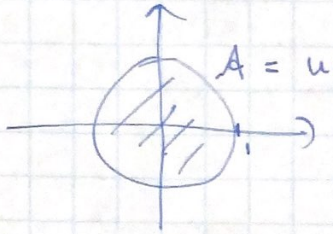
$$\sum_{a \in \Sigma} p(a)\, u_a$$

where $\{u_a : a \in \Sigma\} \subseteq A$, p prob vector.

Def: Convex Hull of a set $A \subseteq \mathcal{X}$, denote $\mathrm{conv}(A)$ is the intersection of all convex sets containing A.

Thm: $\mathrm{conv}(A)$ = convex combinations of points in A.

eg. $X = \mathbb{C}$

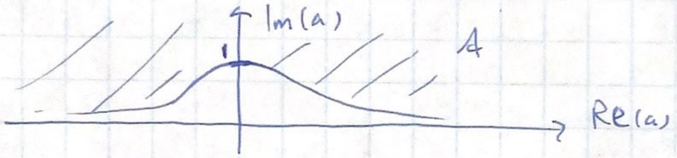


note A infinite but conv(A) is the entire disc; each interior point is convex combination of 2 extreme points.

Thm If A is compact, then conv(A) is compact.

NB If A is closed, conv(A) may or may not be closed.

eg. $X = \mathbb{C}$, $A = \{ a \in \mathbb{C} : \mathrm{Im}(a) \geq \frac{1}{1+[\mathrm{Re}(a)]^2} \}$



Thm (Krein-Milman) : if A compact and convex,

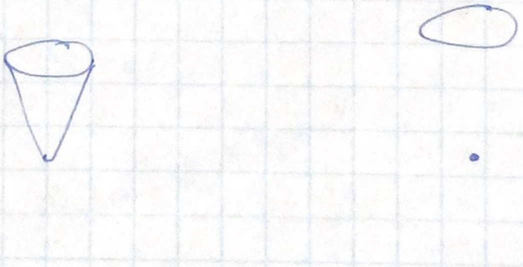
then A = convex hull of its extreme points.

eg. the convex cone ≠ convex hull of its extreme points.

↑

Closed
convex
not bound

the truncated cone is convex hull of its extreme points.

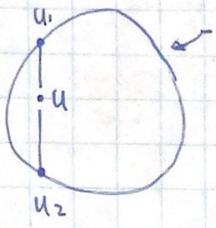


Three thms on convex sets in real Euclidean spaces ($\mathbb{R}^K \sim \mathbb{R}^{|\Sigma|}$):

- Thm 2.6. (Carathéodory's theorem)

If $A \subseteq \mathbb{R}^K$, $u \in \mathrm{conv}(A)$, then $u = \sum_{a=1}^{K+1} p(a)\, u_a$ where $\{u_a\}_{a=1}^{K+1} \subseteq A$.

eg. $K=2$

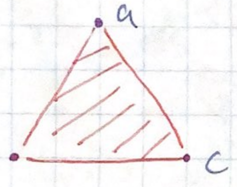


A = unit circle, $\mathrm{conv}(A)$ = unit disc

Each u = conv comb of $K=2$ points in A.

(better than CT)

eg. $K=2$, $A = \{a, b, c\}$, $\mathrm{conv}(A)$ = triangle.



Each interior point is a convex combination of 3 points in A.

- Thm 2.7 (Sion's minmax theorem)

If $A, B \subset \mathbb{R}^K$ are compact and convex

then $\min_{u \in A} \max_{v \in B} \langle u, v \rangle = \max_{v \in B} \min_{u \in A} \langle u, v \rangle$

- Thm 2.8 (Separating hyperplane theorem)

If $A \subseteq \mathbb{R}^K$ is closed and convex, and $u \in \mathbb{R}^K \setminus A$,

then $\exists\, v \in \mathbb{R}^K$ s.t. $\langle v, w \rangle > \langle v, u \rangle$ for all $w \in A$.