QIC 820 / CO781 / CO486 / CS867

Lecture notes for Oct 19: Apps of SDP in Q Info.

1. Q state discrimination problem:

- Alice chooses k wp p_k, creates ρ_k and sends to Bob.
- Bob wants to know k, measures ρ_k and gets outcome i.

Given: $p_k, \rho_k \quad (k = 1, 2, \ldots, n)$

Variable: POVM $(M_1, M_2, \ldots, M_n)$

$$\sum_{k=1}^{n} M_k = \mathbb{1}_X, \quad M_1, \ldots, M_n \in \mathrm{Pos}(X).$$

Goal: sup prob (outcome k = index i)

$$= \sum_k \underbrace{p_k}_{\substack{\text{prob state} \\ \text{is } \rho_k}} \underbrace{\langle M_k, \rho_k \rangle}_{\substack{\text{prob outcome is } k \\ \text{conditioned on state being } \rho_k}}$$

Primal:

$$\alpha = \sup \sum_{k=1}^{n} p_k \langle M_k, \rho_k \rangle$$

$$\text{s.t.} \quad \sum_{k=1}^{n} M_k = \mathbb{1}_X$$

$$M_1, M_2, \ldots, M_n \geq 0$$

in $L(\mathbb{C}^n \otimes X)$

$$\text{Let } X = \begin{bmatrix} M_1 & & * \\ & M_2 & \\ & & \ddots & \\ * & & & M_n \end{bmatrix}, \quad A = \begin{bmatrix} p_1\rho_1 & 0 & 0 & \\ 0 & p_2\rho_2 & & \\ 0 & 0 & \ddots & \\ & & & p_n\rho_n \end{bmatrix} = \sum_k |k\rangle\langle k| \otimes p_k \rho_k, \quad B = \mathbb{1}_X$$

unrestricted.

$$\sum_{k=1}^{n} M_k = \mathrm{tr}_{\mathbb{C}^n} X = \Phi(X) = \sum_{i=1}^{n} (\langle i| \otimes \mathbb{1}_X)\, X\, (|i\rangle \otimes \mathbb{1}_X).$$

$\forall Y \in \mathrm{Herm}(\mathcal{X})$

$$\Phi^*(Y) = \sum_{i=1}^{n} (\underbrace{|i\rangle}_{L(\mathbb{C},\mathbb{C}^n)} \otimes \underbrace{\mathbb{1}_{\mathcal{X}}}_{L(\mathcal{X})})\, \underbrace{Y}_{L(\mathcal{X})}\, (\underbrace{\langle i|}_{L(\mathbb{C}^n,\mathbb{C})} \otimes \underbrace{\mathbb{1}_{\mathcal{X}}}_{L(\mathcal{X})})$$

$$= \sum_{i=1}^{n} |i\rangle\langle i| \otimes \mathbb{1}_{\mathcal{X}} Y \mathbb{1}_{\mathcal{X}}$$

$$= \mathbb{1}_{\mathbb{C}^n} \otimes Y.$$

Alternatively, $\forall Y \in \mathrm{Herm}(\mathcal{X})$,

$$\langle \Phi^*(Y), X\rangle = \langle Y, \Phi(X)\rangle = \langle Y, \mathrm{tr}_{\mathbb{C}^n} X\rangle = \langle \mathbb{1}_{\mathbb{C}^n}\otimes Y, X\rangle$$

$$\therefore \Phi^*(Y) = \mathbb{1}_{\mathbb{C}^n} \otimes Y = \begin{pmatrix} \boxed{Y} & & & & 0 \\ & \boxed{Y} & & & \\ & & \boxed{Y} & & \\ & & & \ddots & \\ 0 & & & & \boxed{Y} \end{pmatrix}$$

n blocks

<u>Primal:</u>

$\sup \langle A, X\rangle$

s.t. $\Phi(X) = \mathrm{tr}_{\mathbb{C}^n} X = \underbrace{\mathbb{1}_{\mathcal{X}}}_{B}$

$X \geq 0$

Dual

$\inf \langle B, Y\rangle = \inf \mathrm{tr}\, Y$

s.t. $\underbrace{\Phi^*(Y) = \mathbb{1}_{\mathbb{C}^n}\otimes Y = \sum_{k=1}^{n} |k\rangle\langle k| \otimes Y \geq A = \sum_{k=1}^{n} |k\rangle\langle k| \otimes p_k \rho_k}$

$\Leftrightarrow \forall k,\ Y \geq p_k \rho_k.$

Theorem 8.3 (from LN2011) Let $\mathcal{M} = (M_1, M_2, \dots, M_n)$ be a measurement.

$\mathcal{M}$ is optimal for discrimination of ensemble $\{(p_i, \rho_i)\}$ $\iff$ ① $Y = \sum_{k=1}^{n} p_k M_k \rho_k \in \mathrm{Herm}(\mathcal{X})$

② $\forall k,\ Y \geq p_k \rho_k$

Pf The SDP is strictly primal feasible

$\because (M_1, M_2, \dots, M_n) = (\frac{1}{n}\mathbb{1}, \dots, \frac{1}{n}\mathbb{1})$ is a strictly feasible sol'n.

The SDP is strictly dual feasible

$\because Y = 2\mathbb{1}$ is a strictly feasible sol'n.

By corollary of Slater's Thm, $\alpha = \beta$, both attained.

[$\Rightarrow$] Suppose $(M_1, \dots, M_n)$ and Y are primal & dual optimal.

By complementary slackness, $X\Phi^*(Y) = XA$, $\Phi^*(Y)X = AX$.

i.e.

$$\begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_n \end{pmatrix} \begin{pmatrix} Y & & & 0 \\ & Y & & \\ & & \ddots & \\ 0 & & & Y \end{pmatrix} = \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_n \end{pmatrix} \begin{pmatrix} p_1\rho_1 & & & \\ & p_2\rho_2 & & \\ & & \ddots & \\ & & & p_n\rho_n \end{pmatrix}$$

or $\forall k = 1, \dots, n$, $\quad M_k Y = M_k p_k \rho_k$ —①

and - - - - $\quad Y M_k = p_k \rho_k M_k$ —② .

From ①, $\underbrace{\sum_k M_k Y}_{=Y} = \sum_k M_k p_k \rho_k$

$\because Y$ feasible, $Y \in \mathrm{Herm}(\mathcal{X})$ $\therefore \sum_k M_k p_k \rho_k \in \mathrm{Herm}(\mathcal{X})$

$\because Y$ feasible, $\Phi^*(Y) \geq A$ $\quad \therefore \forall k, Y \geq p_k \rho_k$

[⇐] If $Y = \sum_{k=1}^{n} p_k M_k \rho_k \in \mathrm{Herm}(X)$

and $\forall k,\ Y \geq p_k \rho_k$

then, Y dual feasible

$$\therefore\ \beta \leq \mathrm{Tr}(Y) = \sum_k p_k \langle M_k, \rho \rangle \overset{M_1 \ldots M_n \text{ legit meas}}{\leq} \alpha \underset{\text{weak duality}}{\leq} \beta$$

$\therefore$ $M_1 \ldots M_n$ optimal measurement.

eg. In A3, you will show that, if

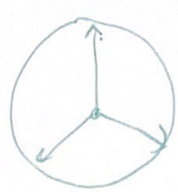
① $p_i = \frac{1}{n}\ \forall i$

② ρ_i pure $\forall i$

③ $\frac{1}{n} \sum_i \rho_i = \frac{\mathbb{1}_X}{\dim X}$

then, $M_i = \frac{\dim X}{n} \rho_i$ gives an optimal measurement!

You might have seen a proof for trine states or other ensembles with a lot of symmetry. But here, none is explicitly needed (except for ③).



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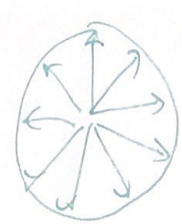

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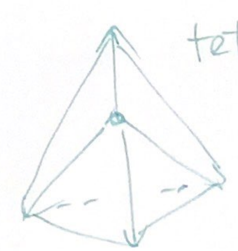

tetrahedron

SIC POVM on $\mathbb{C}^2$

2. Rederiving Helstrom's bound (special case to discriminate 2 states)

Recall $\Pi_{\pm}$ = projection onto $\pm$ve eigenspace of $p_1\rho_1 - p_2\rho_2$

* We now give an alternative proof, using complementary slackness, that $M_1 = \Pi_+$, $M_2 = \Pi_-$ is an optimal measurement.

Using Thm 8.3, consider $Y = p_1 \Pi_+ \rho_1 + p_2 \Pi_- \rho_2$

$$= p_1 \Pi_+ \rho_1 + p_2 (I - \Pi_+) \rho_2$$

$$= \underbrace{\Pi_+ (p_1\rho_1 - p_2\rho_2)}_{+\text{ve part of } p_1\rho_1 - p_2\rho_2} + p_2\rho_2 \quad \in \mathrm{Herm}(\mathcal{X})$$

Also, $Y \geq p_2\rho_2$

Similarly $Y \geq p_1\rho_1$ $\quad \therefore (\Pi_+, \Pi_-)$ is optimal.

3. Setting $p_1 = p_2$ above, and using Helstrom's bdd,

best discrimination prob $= \frac{1}{2} + \frac{1}{4} \|\rho_1 - \rho_2\|_1$

$$= \sup \tfrac{1}{2}\left(\langle M_1, \rho_1\rangle + \langle M_2, \rho_2\rangle\right)$$
$$\text{s.t. } M_1 + M_2 = \mathbb{1}_{\mathcal{X}}$$
$$M_1, M_2 \geq 0$$

This gives an SDP formulation for trace distance.

4 SDP formulation of fidelity & rederiving Alberti's Thm.

Let $P, Q \in \mathrm{Pos}(X)$

Recall $F(P,Q) = \| \sqrt{P}\sqrt{Q} \|_1$

$$\|M\|_1 = \max_{U \text{ unitary}} \{ |\langle U, M\rangle| \} = \max_{\substack{V \text{ contraction} \\ \|V\|_\infty \leq 1}} \{ |\langle V, M\rangle| \}$$

Claims: (1) following SDPs are primal-dual pair

(2) $\alpha = F(P,Q)$

(3) strong duality holds

(4) $\beta = F(P,Q)$

(5) Can derive Alberti's Thm from dual SDP

Primal

$$\alpha = \sup \tfrac{1}{2}\mathrm{Tr}(X) + \tfrac{1}{2}\mathrm{Tr}(X^*)$$

$$\text{s.t.} \begin{bmatrix} P & X \\ X^* & Q \end{bmatrix} \geq 0$$

Dual

$$\beta = \inf \tfrac{1}{2}\langle P, Y\rangle + \tfrac{1}{2}\langle Q, Z\rangle$$

$$\text{s.t.} \begin{bmatrix} Y & -\mathbb{1} \\ -\mathbb{1} & Z \end{bmatrix} \geq 0$$

WLOG, $P, Q \in \mathrm{Pd}(X)$.
If not, take $P + \varepsilon \mathbb{1}_X$ as $\varepsilon \to 0$

Claim 1:

Rewriting the primal constraints:

$$\begin{bmatrix} P & X \\ X^* & Q \end{bmatrix} \in \mathrm{Pos}(\mathcal{X} \oplus \mathcal{X}) \qquad (\mathcal{X} \oplus \mathcal{X} \cong \mathbb{C}^{\{1,2\}} \otimes \mathcal{X})$$

$$\Leftrightarrow \quad K = \begin{bmatrix} X_1 & X \\ X^* & X_2 \end{bmatrix} \in \mathrm{Pos}(\mathcal{X} \oplus \mathcal{X})$$

$$\text{and} \quad \Phi(K) = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

Rewriting the primal objective function:

$$\frac{1}{2}\mathrm{Tr}(X) + \frac{1}{2}\mathrm{Tr}(X^*) = \langle A, K \rangle$$

Where $A = \begin{bmatrix} 0 & \mathbb{1}_{\mathcal{X}} \\ \mathbb{1}_{\mathcal{X}} & 0 \end{bmatrix} \frac{1}{2}$

$\therefore$ Primal SDP:

$$\sup \langle A, K \rangle$$
$$\text{s.t. } \Phi(K) = B = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$
$$K \geq 0$$

Dual SDP:

$$\inf \langle B, L \rangle$$
$$\text{s.t. } \Phi^*(L) \geq A$$
$$L \in \mathrm{Herm}(\mathcal{X} \oplus \mathcal{X})$$

Let $L = \begin{bmatrix} W_1 & W \\ W^* & W_2 \end{bmatrix}$

$\forall K \forall L \quad \langle \Phi^*(L), K \rangle = \langle L, \Phi(K) \rangle$

$$= \left\langle \begin{bmatrix} W_1 & W \\ W^* & W_2 \end{bmatrix}, \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \begin{bmatrix} X_1 & X \\ X^* & X_2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, K \right\rangle$$

$$\therefore \Phi^*(L) = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}.$$

Dual constraint $\Phi^*(L) \geq A$

Same as $\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \geq \frac{1}{2}\begin{bmatrix} 0 & \mathbb{1}_{\mathcal{X}} \\ \mathbb{1}_{\mathcal{X}} & 0 \end{bmatrix}$

Let $Y = \frac{1}{2}W_1$, $Z = \frac{1}{2}W_2$

Dual constraint : $\begin{bmatrix} Y & -\mathbb{1} \\ -\mathbb{1} & Z \end{bmatrix} \geq 0$.

$$\text{Dual obj fun} = \langle B, L \rangle = \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}, \begin{bmatrix} W_1 & W \\ W^* & W_2 \end{bmatrix} \right\rangle$$

$$= \langle P, W_1 \rangle + \langle Q, W_2 \rangle$$

$$= \frac{1}{2}\langle P, Y \rangle + \frac{1}{2}\langle Q, Z \rangle.$$

Claim 2 :

Fun fact : If $C \in \mathrm{Pd}(\mathcal{X})$ $(C > 0)$

then $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathrm{Pos}(\mathcal{X} \oplus \mathcal{X}) \iff \underbrace{A \geq B^* C^{-1} B}_{\text{Schur complement}}$

Pf : internet search ...

(a) $\begin{bmatrix} P & X \\ X^* & Q \end{bmatrix} \geq 0 \iff P \geq X^* Q^{-1} X$

$$\iff \mathbb{1} \geq \underbrace{P^{-\frac{1}{2}} X^* Q^{-\frac{1}{2}}}_{V^*} \underbrace{Q^{-\frac{1}{2}} X P^{-\frac{1}{2}}}_{V} \qquad \text{(need } Q>0,\ P\geq 0\text{)}$$

$$\iff \|V\|_\infty \leq 1$$

(b) $X = Q^{\frac{1}{2}} V P^{\frac{1}{2}}$

$\operatorname{tr} X = \operatorname{tr}(V P^{\frac{1}{2}} Q^{\frac{1}{2}})$ cyclic property of tr

$= \langle V^*, P^{\frac{1}{2}} Q^{\frac{1}{2}} \rangle$

$$\sup_{\|V^*\|_\infty \leq 1} |\operatorname{tr} X| = F(P,Q)$$

(Replace V^* by $V^* e^{i\theta}$ s.t. $|\operatorname{tr} X| = \operatorname{tr} X = \operatorname{tr} X^*$)

$\therefore\ \alpha = F(P,Q)$.

Claim (3):

- $X=0$ gives a feasible primal sol'n $\Rightarrow \beta$ bounded
- $Y=Z=2\mathbb{1}$ gives strictly feasible dual sol'n

Slater's thm for the dual $\Rightarrow \alpha = \beta$, α is attained

Claim (4): follows from claims (2) (3)

Claim (5).

Dual constraint $\begin{bmatrix} Y & -\mathbb{1} \\ -\mathbb{1} & Z \end{bmatrix} \geq 0 \iff Y \geq Z^{-1}$ (need $Z \in \mathrm{Pd}(X)$)

Obj function $= \frac{1}{2}\langle P, Y\rangle + \frac{1}{2}\langle Q, Z\rangle \geq \frac{1}{2}\langle P, Z^{-1}\rangle + \frac{1}{2}\langle Q, Z\rangle$

$\therefore \beta = \inf\limits_{Z \in \mathrm{Pd}(X)} (\text{obj fun}) = \inf\limits_{Z \in \mathrm{Pd}(X)} \frac{1}{2}\langle P, Z^{-1}\rangle + \frac{1}{2}\langle Q, Z\rangle$

For each Z, $\frac{1}{2}\langle P, Z^{-1}\rangle + \frac{1}{2}\langle Q, Z\rangle$

$$\geq \sqrt{\langle P, Z^{-1}\rangle\langle Q, Z\rangle} \quad (\text{AM} \geq \text{GM})$$

$$= \sqrt{\langle P, (Z\lambda)^{-1}\rangle \langle Q, \lambda Z\rangle}$$ ← choose λ s.t. 2 inner products equal

$$= \frac{1}{2}\left(\langle P, (Z\lambda)^{-1}\rangle + \langle Q, \lambda Z\rangle\right)$$

$\therefore \beta = \inf\limits_{\substack{\tilde{Z} \in \mathrm{Pd}(X) \\ \parallel \\ \lambda Z}} \sqrt{\langle P, \tilde{Z}^{-1}\rangle \cdot \langle Q, \tilde{Z}\rangle} = F(P, Q)$

$$\boxed{\inf_{\tilde{Z} > 0} \langle P, \tilde{Z}^{-1}\rangle \langle Q, \tilde{Z}\rangle = F^2(P, Q)}$$

Alberti's Thm