QIC 820 / CO781 / 486 / CS 867 Part 3 lecture 4

Highlights of results in lectures 10-12 in LN 2011

- Thm 10.5 X = CES, $\rho, \xi \in D(X)$, $\|\rho-\xi\|_1 \leq 1/e$

  Then $|S(\rho) - S(\xi)| \leq \log(\dim(X)) \cdot \|\rho-\xi\|_1 + \frac{1}{\ln 2}\,\eta(\|\rho-\xi\|_1)$

  where $\eta(x) = -x \ln x$.

- Def $P, Q \in Pd(X)$, $S(P\|Q) = \mathrm{Tr}\left[P(\log P - \log Q)\right]$.

- Thm 10.6 $\rho, \xi \in D(X) \cap Pd(X)$, $S(\rho\|\xi) \geq \frac{1}{2\ln 2}\|\rho-\xi\|_2^2$

  if 1, Pinsker's ineq

Last time

This time

- Cor 10.7 $\forall \rho \in D(X)$, $0 \overset{①}{\leq} S(\rho) \overset{②}{\leq} \log(\dim(X))$

  NB: $S(\rho) \leq \log(\mathrm{rank}(\rho))$

  Equality holds $\forall$ pure state ρ for ①

  - - - - - - for $\rho = \frac{I}{\dim(X)}$ only for ②.

- Def: for $\rho \in D(X \otimes Y)$

$$S(XY) = S(\rho), \quad S(X) = S(\rho^X), \quad S(Y) = S(\rho^Y)$$

$$\rho^X = \mathrm{Tr}_Y(\rho), \qquad \rho^Y = \mathrm{tr}_X(\rho)$$

- Thm 10.8 Subadditivity of VN entropy: $\forall \rho \in D(X \otimes Y)$, $S(XY) \leq S(X) + S(Y)$.

  Pf: $S(\rho^{XY} \| \rho^X \otimes \rho^Y)$ "how far ρ is from the product of the marginals"

$$= -S(\rho^{XY}) + S(\rho^X) + S(\rho^Y) \quad (=: S(X:Y)_\rho \text{ Quantum mutual info})$$

  Sometime we say what state we are evaluating the entropies on.

- Subadd follows from Thm 10.6.

- $S(XY) = S(X) + S(Y) \iff \rho = \rho^X \otimes \rho^Y$. (Also from 10.6)

  ie QMI $= 0 \iff \rho$ product op ie X, Y indep.

Thm 10.9 Concavity of vN entropy

If $\rho, \xi \in D(\mathcal{X})$, $\lambda \in [0,1]$, then $S(\lambda\rho + (1-\lambda)\xi) \geq \lambda S(\rho) + (1-\lambda) S(\xi)$.

ie mixing ↑ entropy

NC00

Thm: entropy of classical-quantum states.

Let $\rho \in D(\mathbb{C}^{\Sigma} \otimes \mathcal{X})$, $\quad \rho = \sum_{a \in \Sigma} p(a) \underbrace{|a\rangle\langle a|}_{\text{basis states for } \mathbb{C}^{\Sigma}} \otimes \rho_a$ ← in $D(\mathcal{X})$

$$\text{Then } S(\rho) = \sum_{a \in \Sigma} p(a) S(\rho_a) + H(p).$$

↑ Shannon entropy of r.v. with sample space Σ, distⁿ $p(a)$.

Def: X, Y r.v. with sample spaces Ω^X, Ω^Y, joint dist $p(xy)$.

Let "$X \mid Y = a$" be r.v with sample space Ω^X

↑ distⁿ of X conditioned on $Y=a$.

prob of outcome $x = \dfrac{p(xa)}{p(a)}$, $\quad p(a) = \sum_{x \in \Omega^X} p(xa)$

$$H(X|Y) = \sum_{a \in \Omega^Y} p(a) \cdot H(X \mid Y = a).$$

Thm (chain rule): $\underbrace{H(XY)}_{\substack{\text{defined on} \\ \text{joint dist}^n \\ p(xy).}} = \underbrace{H(Y)}_{\substack{\text{defined on} \\ p(a)}} + \underbrace{H(X|Y)}_{\substack{\text{defined} \\ \text{above.}}}$

sym in X, Y

Def: $I(X:Y) = H(X) - H(X|Y) = H(X) + H(Y) - H(XY)$.

reduction in ignorance of X given Y

<u>Def</u> $\rho \in D(\mathcal{X} \otimes \mathcal{Y})$, $S(X|Y) := S(XY) - S(Y)$.

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definition emulating $H(X|Y)$'s chain rule.

<u>Thm.</u> if $\rho \in D(\mathcal{X} \otimes \mathcal{Y})$, $\mathcal{Y} \cong \mathbb{C}^{\Sigma}$,

and $\rho = \sum_{a \in \Sigma} p(a)\, \rho_a \otimes |a\rangle\langle a|$,

then $S(X|Y)_\rho = \sum_{a \in \Sigma} p(a)\, S(\rho_a)$.

Pf: $S(\rho) = S(XY) = \underbrace{H(p)}_{S(Y)} + \underbrace{\sum_{a \in \Sigma} p(a)\, S(\rho_a)}_{\therefore \text{ This is } S(X|Y).}$ from Thm^{NC}.

Thm is analoguous to def of $H(X|Y)$, but <u>only holds for ρ being Q-C with $\mathcal{Y}$ classical</u> !!

- Alt def of $S(X:Y) = S(X) - S(X|Y)$ (instead of $S(\rho^{XY} \| \rho^X \otimes \rho^Y)$)

$= S(X) + S(Y) - S(XY)$.

<u>Thm:</u> $S(\rho) = S(U \rho U^*)$

$S(X:Y)_\rho = S(X:Y)_{U_X \otimes V_Y \rho U_X^* \otimes V_Y^*}$

$S(\rho \| \sigma) = S(U\rho U^* \| U \sigma U^*)$

<u>Thm:</u> $S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$

(from pf of Thm 10.8)

A collection of very important results related to <u>strong subadditivity</u> SSA of vN entropy

(1) Thm 11.2 Joint convexity of the quantum relative entropy.

K CES, $\rho_0, \rho_1, \sigma_0, \sigma_1 \in D(X) \cap Pd(X)$, $\lambda \in [0,1]$

Then $S(\lambda \rho_0 + (1-\lambda)\rho_1 \| \lambda \sigma_0 + (1-\lambda)\sigma_1) \leq \lambda S(\rho_0 \| \sigma_0) + (1-\lambda) S(\rho_1 \| \sigma_1)$

Interpretation: mixing decreases distinguishability (measured by QRE).

(2) Cor 11.7 $\forall \rho, \sigma \in D(X \otimes Y) \cap Pd(X \otimes Y)$

$$S(\mathrm{Tr}_Y(\rho) \| \mathrm{Tr}_Y(\sigma)) \leq S(\rho \| \sigma)$$

Pf sketch: from LN 2011 Ch 6: completely depolarizing channel on Y: $\Lambda_Y(M) = \frac{\mathbb{1}_Y}{\dim(Y)}$ $\forall M \in L(Y)$.

(a) $\Lambda_Y(M) = \frac{1}{\dim(Y)^2} \sum_k U_k M U_k^\dagger$ ← Weyl operators.

(b) $\rho \in D(X \otimes Y)$, $I_X \otimes \Lambda_Y(\rho) = \mathrm{tr}_Y(\rho) \otimes \frac{\mathbb{1}_Y}{\dim Y} = \frac{1}{(\dim Y)^2} \sum_k \mathbb{1}_X \otimes U_k \, \rho \, \mathbb{1}_X \otimes U_k^*$

$\therefore S(\mathrm{Tr}_Y(\rho) \| \mathrm{Tr}_Y(\sigma))$

$= S\left(\mathrm{Tr}_Y(\rho) \otimes \frac{\mathbb{1}_Y}{\dim Y} \| \mathrm{Tr}_Y(\sigma) \otimes \frac{\mathbb{1}_Y}{\dim Y}\right)$

Ex: $S(P \otimes R \| Q \otimes R) = S(P \| Q)$ $\forall P, Q, R \in Pd$

$= S\left(\frac{1}{(\dim Y)^2} \sum_k (\mathbb{1}_X \otimes U_k)\rho(\mathbb{1}_X \otimes U_k^*) \| \frac{1}{(\dim Y)^2} \sum_k (\mathbb{1}_X \otimes U_k)\sigma(\mathbb{1}_X \otimes U_k^*)\right)$.

$$\overset{\text{Th 11.2}}{\leq} \frac{1}{(\dim Y)^2} \sum_k \underbrace{S((\mathbb{1}_X \otimes U_k)\rho(\mathbb{1}_X \otimes U_k^*) \| (\mathbb{1}_X \otimes U_k)\sigma(\mathbb{1}_X \otimes U_k^*))}_{S(\rho\|\sigma) \text{ for each } k.}$$

$= S(\rho \| \sigma)$

(3) Cor 11.8 $\forall \rho, \sigma \in D(X),\ \Phi \in C(X, Y)$

$$S(\Phi(\rho) \| \Phi(\sigma)) \leq S(\rho \| \sigma)$$

Interpretation: processing by Φ cannot increase distinguishability meas by QRE

Pf Sketch: Φ is composition of an isometry & partial trace.

leaves QRE inv; cannot increase QRE

(4) Thm 11.1 Strong subadd (SSA) of vN entropy

$$\forall \rho \in D(X \otimes Y \otimes Z), \quad S(XYZ) + S(Z) \leq S(XZ) + S(YZ)$$

NB: If $Z = \mathbb{C}$, SSA becomes SA.

Pf sketch: routine to check

$$S\left(\rho^{XYZ} \,\Big\|\, \frac{\mathbb{1}_X}{\dim X} \otimes \rho^{YZ}\right) = -S(\rho^{XYZ}) + S(\rho^{YZ}) + \log(\dim X)$$

$$\underset{\text{Cor 11.7}}{\geq} S\left(\rho^{XZ} \,\Big\|\, \frac{\mathbb{1}_X}{\dim X} \otimes \rho^{Z}\right) = -S(\rho^{XZ}) + S(\rho^{Z}) + \log(\dim X)$$

$\therefore\ S(XYZ) + S(Z) \leq S(YZ) + S(XZ)$

(5) Cor 11.9 Mono of QMI wrt partial tracing.

$$\forall \rho \in D(X \otimes Y \otimes Z), \quad S(X:Z) \leq S(X:YZ)$$

Pf sketch: LHS $= S(X) + S(Z) - S(XZ)$

RHS $= S(X) + S(YZ) - S(XYZ)$.

$\therefore$ RHS $\geq$ LHS by SSA

(6) Cor: $S(X:Z)_{I \otimes \Phi(\rho)} \leq S(X:Y)_\rho \qquad \forall \rho \in D(X \otimes Y),\ \Phi \in C(Y, Z)$