QIC 820 / CO781 / 486 / CS867 Part 4 lec 1

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For bipartite pure states, it is intuitive how to distinguish entangled states from otherwise.

Entangled	Not entangled
① $\lvert\phi\rangle = \sum_{i=1}^{K} c_i \lvert\alpha_i\rangle_A \otimes \lvert\beta_i\rangle_B$	$\lvert\psi\rangle = \lvert\alpha\rangle_A \otimes \lvert\beta\rangle_B$ tensor product
Schmidt decomposition with $c_i > 0$, $\{\lvert\alpha_i\rangle\}$ onset, $\{\lvert\beta_i\rangle\}$ onset, $K > 1$ (Schmidt rank)	Schmidt rank = 1
② $S(\mathrm{tr}_B \lvert\phi\rangle\langle\phi\rvert) > 0$	$S(\mathrm{tr}_B \lvert\phi\rangle\langle\phi\rvert) = 0$
Entropy of entanglement - # ebits extractable per copy - # ebits required to produce	
③ Bell ineq violation	Hidden variable models

Much more complex for mixed states ...

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Def (separable operator):

Let $\mathcal{X}, \mathcal{Y}$ be CES, $P \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$

P is separable $\iff$ $\exists\, Q_1 \dots Q_m \in \mathrm{Pos}(\mathcal{X})$, $R_1, \dots R_m \in \mathrm{Pos}(\mathcal{Y})$ s.t $P = \sum_{j=1}^{m} Q_j \otimes R_j$

Def: set of all sep operators on $\mathcal{X}, \mathcal{Y}$: $\mathrm{Sep}(\mathcal{X} : \mathcal{Y})$

set of all sep density ops on $\mathcal{X}, \mathcal{Y}$: $\mathrm{SepD}(\mathcal{X} : \mathcal{Y}) = \mathrm{Sep}(\mathcal{X} : \mathcal{Y}) \cap D(\mathcal{X} \otimes \mathcal{Y})$

set of all entangled operators on $\mathcal{X}, \mathcal{Y}$ = $\mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y}) \setminus \mathrm{Sep}(\mathcal{X} : \mathcal{Y})$

Facts (require simple proofs left as Ex)

① $\mathrm{Sep}(\mathcal{X} : \mathcal{Y})$ is a convex cone, closed

② $\mathrm{SepD}(\mathcal{X} : \mathcal{Y})$ is convex & compact

Thm ⓐ $\mathrm{SepD}(\mathcal{X} : \mathcal{Y}) = \mathrm{conv}\{ xx^* \otimes yy^* : x \in S(\mathcal{X}),\ y \in S(\mathcal{Y}) \}$

ⓑ If $\rho \in \mathrm{SepD}(\mathcal{X} : \mathcal{Y})$, then $\rho = \sum_{i=1}^{m} p(i)\, x_i x_i^* \otimes y_i y_i^*$

for some prob vec $(p(1), p(2), \dots, p(m))$

$m \leq \dim(\mathcal{X} \otimes \mathcal{Y})^2$

$x_1 \dots x_m \in S(\mathcal{X})$, $y_1, \dots, y_m \in S(\mathcal{Y})$

Pf: ⓐ Clearly $\mathrm{SepD}(\mathcal{X} : \mathcal{Y}) \supseteq \mathrm{conv}\{ xx^* \otimes yy^* : x \in S(\mathcal{X}),\ y \in S(\mathcal{Y}) \}$

Conversely, if (i) $\rho \in \mathrm{Sep}(\mathcal{X} : \mathcal{Y})$, $\rho = \sum_{j=1}^{n} Q_j \otimes R_j$, $Q_j \in \mathrm{Pos}(\mathcal{X})$, $R_j \in \mathrm{Pos}(\mathcal{Y})$

Let $Q_j = \sum_{l} q_{jl}\, u_{jl} u_{jl}^*$, $R_j = \sum_{k} t_{jk}\, v_{jk} v_{jk}^*$ be spec decomps

with $q_{jl}, t_{jk} \geq 0$, $u_{jl} \in S(\mathcal{X})$, $v_{jk} \in S(\mathcal{Y})$

Then $\rho = \sum_{j=1}^{n} \sum_{l} \sum_{k} q_{jl} t_{jk} \, u_{jl} u_{jl}^* \otimes v_{jk} v_{jk}^*$

(ii) if $\rho \in D(\mathcal{X} \otimes \mathcal{Y})$, $\mathrm{Tr}\,\rho = 1 = \sum_{j=1}^{n} \sum_{l} \sum_{k} q_{jl} t_{jk}$.

$\therefore \rho \in \mathrm{SepD}(\mathcal{X} : \mathcal{Y}) \Rightarrow \rho \in \mathrm{conv}\{ x x^* \otimes y y^* : x \in S(\mathcal{X}), y \in S(\mathcal{Y}) \}$

(b)

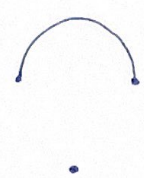
We first state Thm 2.6:

Carathéodory's Thm: if $A \subseteq \mathbb{R}^n$, then $\forall u \in \mathrm{conv}(A)$

$u = \sum_{i=1}^{n+1} p_i u_i$ for some $u_1, \dots, u_{n+1} \in A$

$\{p_i\}$ dist^n over $\{1, \dots, n+1\}$

eg. $A \subseteq \mathbb{R}^2$,

A: 

conv (A) 

Pf (b): $x x^* \otimes y y^* \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y}) \subseteq \mathrm{Herm}(\mathcal{X} \otimes \mathcal{Y}) \cong \mathbb{R}^{\dim(\mathcal{X} \otimes \mathcal{Y})^2}$

but $x x^* \otimes y y^*$ trace 1 $\therefore$ $x x^* \otimes y y^*$ lives in a $\dim(\mathcal{X} \otimes \mathcal{Y})^2 - 1$ dim real Euclidean space.

$\therefore$ Carathéodory's theorem $\Rightarrow$ $\rho \in \mathrm{SepD}(\mathcal{X} : \mathcal{Y}) = \mathrm{conv}\{ x x^* \otimes y y^* \}$

$$\rho = \sum_{i=1}^{m} p_i \, x_i x_i^* \otimes y_i y_i^*$$

for $m \leq (\dim(\mathcal{X} \otimes \mathcal{Y}))^2 - 1 + 1$

NB Any $\rho \in \mathrm{SepD}(\mathcal{X} : \mathcal{Y})$ can be prepared by local operations on $\mathcal{X}$, $\mathcal{Y}$ augmented with shared randomness / classical communication, ie needs no Q comm nor entanglement to prepared.

Fact: $\mathcal{X}$ CES, $\mathcal{A} \subseteq \text{Herm}(\mathcal{X})$ closed convex cone, $B \in \text{Herm}(\mathcal{X}) \setminus \mathcal{A}$

then $\exists H \in \text{Herm}(\mathcal{X})$ s.t. 1. $\langle H, A \rangle \geq 0 \quad \forall A \in \mathcal{A}$

2. $\langle H, B \rangle < 0$

NB This is similar to the Hyperplane separation theorem.

With $\mathcal{A}$ not just convex but a closed cone and with $c=0$

Thm 14.1 (Horodecki criterion)

$\mathcal{X}, \mathcal{Y}$: CES, $P \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y})$

① $P \in \text{Sep}(\mathcal{X} : \mathcal{Y}) \Leftrightarrow \forall$ positive $\Phi \in T(\mathcal{X}, \mathcal{Y})$ ②

$\Phi \otimes I_{\mathcal{Y}}(P) \in \text{Pos}(\mathcal{Y} \otimes \mathcal{Y})$

$\Leftrightarrow \forall$ positive, unital $\Phi \in T(\mathcal{X}, \mathcal{Y})$ ③

$\Phi \otimes I_{\mathcal{Y}}(P) \in \text{Pos}(\mathcal{Y} \otimes \mathcal{Y})$

Pf: [① ⇒ ②]

If $P = \sum_{j=1}^{m} Q_j \otimes R_j$ for some $Q_1, \ldots, Q_m \in \text{Pos}(\mathcal{X})$, $R_1, \ldots, R_m \in \text{Pos}(\mathcal{Y})$

then $\forall$ positive $\Phi \in T(\mathcal{X}, \mathcal{Y})$, $\Phi(Q_1), \ldots \Phi(Q_m) \in \text{Pos}(\mathcal{Y})$

$\therefore \Phi \otimes I_{\mathcal{Y}}(P) = \sum_{j=1}^{m} \Phi(Q_j) \otimes R_j \in \text{Pos}(\mathcal{Y} \otimes \mathcal{Y})$

Note ② ⇒ ③ is immediate.

[③ ⇒ ①] or [¬① ⇒ ¬③]:

If $P \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y}) \setminus \mathrm{Sep}(\mathcal{X} : \mathcal{Y})$,

apply "Fact" with $A = \mathrm{Sep}(\mathcal{X} : \mathcal{Y})$, $B = P$

$\therefore \exists H \in \mathrm{Herm}(\mathcal{X} \otimes \mathcal{Y})$ s.t. 1. $\forall Q \in \mathrm{Pos}(\mathcal{X}), R \in \mathrm{Pos}(\mathcal{Y}), \langle H, Q \otimes R \rangle \geq 0$

2. $\langle H, P \rangle < 0$

Let $\Psi \in T(\mathcal{Y}, \mathcal{X})$ s.t. $J(\Psi) = H$. (Recall from reps of channels that Ψ exists and is unique.)

Choi matrix

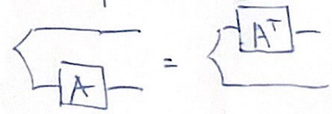
Let $\beta = \sum_{i=1}^{\dim(\mathcal{Y})} e_i \otimes e_i$ $\therefore$ $H = (\Psi \otimes I)(\beta \beta^*)$.

Rephrasing 1:

$\forall Q \in \mathrm{Pos}(\mathcal{X}), R \in \mathrm{Pos}(\mathcal{Y})$

$$
\begin{aligned}
0 &\leq \langle (\Psi \otimes I)(\beta \beta^*), Q \otimes R \rangle \\
&= \langle \beta \beta^*, \Psi^*(Q) \otimes R \rangle \\
&= \mathrm{Tr}\left(I \otimes \sqrt{R}\ \beta \beta^*\ I \otimes \sqrt{R} \right) \cdot \left(\Psi^*(Q) \otimes \mathbb{1}_{\mathcal{Y}} \right) \\
&= \mathrm{tr}\left(\sqrt{R}^T \otimes I\ \beta \beta^*\ \sqrt{R}^T \otimes I \right) \cdot \left(\Psi^*(Q) \otimes \mathbb{1}_{\mathcal{Y}} \right) \\
&= \mathrm{tr}\left(\beta \beta^* \cdot \sqrt{R}^T \Psi^*(Q) \sqrt{R}^T \otimes \mathbb{1}_{\mathcal{Y}} \right) \\
&= \mathrm{tr}\left[\left(\mathrm{Tr}_2\, \beta \beta^* \right) \cdot \sqrt{R}^T \Psi^*(Q) \sqrt{R}^T \right] \\
&= \mathrm{Tr}\left[\mathbb{1}_{\mathcal{Y}} \cdot \sqrt{R}^T \Psi^*(Q) \sqrt{R}^T \right] \\
&= \mathrm{tr}\, \Psi^*(Q) \cdot R^T
\end{aligned}
$$

Transpose trick



$\because\ 0 \leq \mathrm{Tr}(\Psi^*(Q) \cdot R^{\top}) \quad \forall R^{\top} \in \mathrm{Pos}(\mathcal{Y})$

$\therefore\ \Psi^*(Q) \in \mathrm{Pos}(\mathcal{Y})$

$\because$ Above holds $\forall Q \in \mathrm{Pos}(\mathcal{X})$ $\therefore\ \Psi^*$ positive map.

Rephrasing 2:

$0 > \langle (\Psi \otimes I)\beta\beta^*, P \rangle$

$= \langle \beta\beta^*, (\Psi^* \otimes I)(P) \rangle$

$\because\ \beta\beta^* \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{Y}),\ (\Psi^* \otimes I)(P) \notin \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{Y})$.

$\Rightarrow \neg ③$

$[\neg ① \Rightarrow \neg ②]$: from the proof for $[\neg ① \Rightarrow \neg ③]$

choose $\varepsilon > 0$ s.t. $\langle H, P \rangle + \varepsilon \mathrm{Tr}(P) < 0$

define $\Xi \in T(\mathcal{X}, \mathcal{Y})$ as $\Xi(A) = \Psi^*(A) + \varepsilon \mathrm{Tr}(A)\, \mathbb{1}_{\mathcal{Y}}$

$\because\ \Xi(\mathbb{1}_{\mathcal{X}}) \in \mathrm{Pd}(\mathcal{Y})$.

define $\Phi(A) = \Xi(\mathbb{1}_{\mathcal{X}})^{-\frac{1}{2}}\ \Xi(A)\ \Xi(\mathbb{1}_{\mathcal{X}})^{-\frac{1}{2}}$

Ψ^* positive $\Rightarrow \Xi$ positive $\Rightarrow \Phi$ positive.

$\Phi(\mathbb{1}_{\mathcal{X}}) = \mathbb{1}_{\mathcal{Y}}$ $\therefore$ unital

$$0 > \langle H, P \rangle + \varepsilon \operatorname{Tr}(P)$$

$$= \langle J(\Phi) + \varepsilon \mathbb{1}_{\mathcal{Y}\otimes\mathcal{X}}, P \rangle$$

$$= \langle J(\Xi^*), P \rangle$$

$$= \langle \Xi^* \otimes I_{\mathcal{Y}}(\beta\beta^*), P \rangle$$

$$= \langle \beta\beta^*, \Xi \otimes I_{\mathcal{Y}}(P) \rangle$$

$$= \langle \beta\beta^*, \left(\Xi(\mathbb{1}_{\mathcal{X}})^{-\frac{1}{2}} \otimes \mathbb{1}_{\mathcal{Y}}\right) \cdot \underbrace{\Phi \otimes I_{\mathcal{Y}}(P)}_{\notin \operatorname{Pos}(\mathcal{Y}\otimes\mathcal{Y})} \cdot \left(\Xi(\mathbb{1}_{\mathcal{X}})^{-\frac{1}{2}} \otimes \mathbb{1}_{\mathcal{Y}}\right) \rangle$$

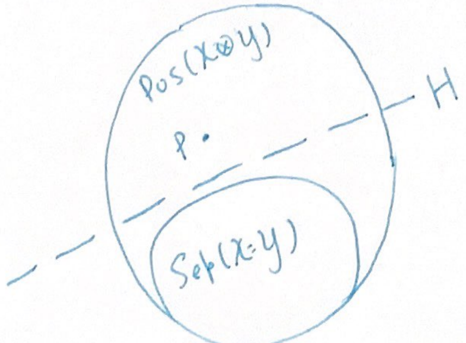
else the last inner product is non neg.

$\therefore$ Φ positive, unital and $\Phi \otimes I_{\mathcal{Y}}(P) \notin \operatorname{Pos}(\mathcal{Y}\otimes\mathcal{Y})$ $\therefore \neg$ (2).

Terminology & intuition:

When P entangled ($\neg$(1)), the operator H in the "fact" or the Φ that refutes (3) are called an entanglement witness for P.

(Given either, you can prove P entangled.)



Somewhat misleading:

eg: extreme points of $\operatorname{Pos}(\mathcal{X}\otimes\mathcal{Y}) \cap \operatorname{Sep}(\mathcal{X}:\mathcal{Y})$ aren't so tightly connected?

eg: it takes $\geq \exp(\text{const} \times d^3 \log d)$ ent witnesses to form a polytope to crudely approx $\operatorname{Sep}(\mathcal{X}:\mathcal{Y})$ (Thm 9.34 Aubrun-Szarek book) eg. for $P = \beta\beta^*$ ($\propto$ max ent state)

consider Φ = transpose map.

ie $\Phi(e_a e_b^*) = e_b e_a^*$ $\forall$ comp basis states e_b, e_a.

(so $e_a^* = e_a^T$, $e_b^* = e_b^T$)

$$\Phi \otimes I (\beta\beta^*)$$

$$= \Phi \otimes I \left(\left(\sum_a e_a \otimes e_a\right)\left(\sum_b e_b^* \otimes e_b^*\right) \right)$$

$$= \Phi \otimes I \quad \sum_{ab} (e_a e_b^*) \otimes (e_a e_b^*)$$

$$= \sum_{ab} \Phi(e_a e_b^*) \otimes (e_a e_b^*)$$

$$= \sum_{ab} e_b e_a^* \otimes e_a e_b^*$$

$$= \sum_{ab} (e_b \otimes e_a) \cdot (e_a^* \otimes e_b^*) \quad = \text{SWAP operator.}$$

takes $e_a \otimes e_b$

outputs $e_b \otimes e_a$.

Take any $a \neq b$, $u = e_a e_b - e_b e_a$ ("singlet", antisym)

then $u^* \; \Phi \otimes I(\beta\beta^*) \, u = u^*(-u) < 0$

$\therefore \Phi \otimes I(\beta\beta^*) \notin \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{Y})$, $\therefore \beta\beta^* \notin \mathrm{Sep}(\mathcal{Y} : \mathcal{Y})$

(here $\mathcal{X} = \mathcal{Y}$).

So we proved that the "max ent state" is entangled. lol...

eg a more refined question:

How much can we mix up the max ent state with the max mixed state (garbage) and the mixture remains entangled?

Ans: Let $\rho_0 = \frac{1}{\dim(\mathcal{Y})} \beta\beta^*$, $\rho_1 = \frac{1}{(\dim \mathcal{Y})^2} \mathbb{1}_{\mathcal{Y}} \otimes \mathbb{1}_{\mathcal{Y}}$

Let $P = \lambda \rho_0 + (1-\lambda)\rho_1$, Φ = transpose map

$$(\Phi \otimes I)(P) = \lambda \cdot \frac{1}{\dim(\mathcal{Y})} \cdot \mathrm{SWAP} + (1-\lambda) \frac{1}{(\dim \mathcal{Y})^2} \mathbb{1}_{\mathcal{Y}} \otimes \mathbb{1}_{\mathcal{Y}}$$

eigenvalues $= \pm 1$
(Hermitian, $\mathrm{SWAP}^2 = 1$)

$$\therefore (\Phi \otimes I)(P) \begin{cases} \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{Y}) & \text{if } \lambda \leq \frac{(1-\lambda)}{\dim(\mathcal{Y})} \text{ or } \lambda \leq \frac{1}{1+\dim(\mathcal{Y})} \\ \notin \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{Y}) & \text{otherwise} \end{cases}$$

- Note $(\Phi \otimes I)(P) \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{Y})$ need not imply $P \in \mathrm{SepD}(\mathcal{Y} : \mathcal{Y})$ since we only check one entanglement witness (Φ = Transpose) here.

- Determining membership of Sep is generally difficult.

- It does hold that P sep for $\lambda \leq \frac{1}{1+\dim(\mathcal{Y})}$ (example 7.25 in book).

14.3 Sep ball around the identity:

Idea: If we mix in enough max mix states,
the final state will have no entanglement (sep).

Need 2 lemmas to prove this.

Lemma 14.2 Let $A \in L(X \otimes \mathbb{C}^{\Sigma})$, $A = \sum_{a,b \in \Sigma} A_{a,b} \otimes (e_a e_b^*)$.

$\uparrow$ in $L(X)$

Then $\|A\|^2 \leq \sum_{a,b \in \Sigma} \|A_{a,b}\|^2$.

Pf: Easier to visualize the problem by swapping the tensor components.

$$A = \sum_{a,b \in \Sigma} e_a e_b^* \otimes A_{a,b} \in L(\mathbb{C}^{\Sigma} \otimes X)$$

$$= \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots \\ A_{2,1} & A_{2,2} & \cdots \\ \vdots & \vdots & \end{bmatrix} \quad (\text{where } \Sigma = \{1, 2, \dots\})$$

$$B_a = \sum_{b \in \Sigma} (e_a e_b^*) \otimes A_{a,b} = \begin{bmatrix} 0 \\ A_{a,1} \;\; A_{a,2} \;\; \cdots \\ 0 \end{bmatrix}$$

$$B_a B_a^* = \left[\sum_{b_1} (e_a e_{b_1}^*) \otimes A_{a,b_1}\right]\left[\sum_{b_2} (e_{b_2} e_a^*) \otimes A_{a,b_2}^*\right] \quad (\text{only } b_1 = b_2 \text{ terms survive}).$$

$$= \sum_b e_a e_a^* \otimes A_{a,b} A_{a,b}^* = e_a e_a^* \otimes \sum_b A_{a,b} A_{a,b}^*$$

$$\begin{bmatrix} 0 \\ A_{a1} \;\; A_{a2} \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \begin{matrix} A_{a1}^* \\ A_{a2}^* \end{matrix} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \square & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\square$ $\sum_b A_{a,b} A_{a,b}^*$ on the (a,a)-block

$$\therefore \; \| B_a B_a^* \| = \Big\| \sum_{b\in\Sigma} A_{a,b} A_{a,b}^* \Big\| \le \sum_{b\in\Sigma} \| A_{a,b} A_{a,b}^* \| = \sum_{b\in\Sigma} \| A_{a,b} \|^2 .$$

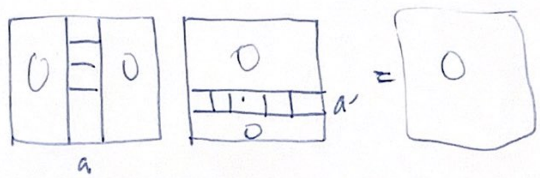
Also $A = \sum_{a\in\Sigma} B_a$, $\quad B_a^* B_{a'} = 0$ if $a \neq a'$

$$\Big[\sum_{b_1} e_{b_1} e_a^* \otimes A_{a,b_1} \Big] \Big[\sum_{b_2} (e_{a'} e_{b_2}^*) \otimes A_{a',b} \Big]$$

$$\shortparallel$$

$$0$$

or



$$\therefore \; A^* A = \sum_{a'\in\Sigma} B_{a'}^* \sum_{a\in\Sigma} B_a = \sum_{a\in\Sigma} B_a^* B_a$$

$$\therefore \; \|A\|^2 = \|A^* A\| = \Big\| \sum_{a\in\Sigma} B_a^* B_a \Big\| \le \sum_{a\in\Sigma} \| B_a^* B_a \|$$

$$= \sum_{a\in\Sigma} \| B_a B_a^* \| \le \sum_{a,b\in\Sigma} \| A_{a,b} \|^2 .$$

<u>Thm 14.3</u> If $\Phi \in T(\mathcal{X}, \mathcal{Y})$ positive & unital,

then $\forall X \in L(\mathcal{X})$, $\| \Phi(X) \| \leq \|X\|$.

Pf: first we show $\| \Phi(U) \| \leq 1$ if U unitary.

By spectral decomp, $U = \sum_{a \in \Sigma} \lambda_a u_a u_a^*$, $|\lambda_a| = 1$

$\Phi(u_a u_a^*) \in Pos(\mathcal{Y})$ since Φ positive

$\sum_a \Phi(u_a u_a^*) = \Phi(\sum_a u_a u_a^*) = \Phi(\mathbb{1}_\mathcal{X}) = \mathbb{1}_\mathcal{Y}$ since Φ unital

$\therefore \{ \Phi(u_a u_a^*) \}_a$ is a POVM.

By Naimark's theorem, $\exists$ isometry $A \in L(\mathcal{Y}, \mathcal{Y} \otimes \mathbb{C}^\Sigma)$ s.t.

$\forall a$, $\Phi(u_a u_a^*) = A^* (\mathbb{1}_\mathcal{Y} \otimes e_a e_a^*) A$

Final, $\Phi(U) = \Phi(\sum_a \lambda_a u_a u_a^*)$

$$= A^* \underbrace{(\mathbb{1}_\mathcal{Y} \otimes \underbrace{\sum_a \lambda_a e_a e_a^*}_{\text{evals}: \lambda_a})}_{\text{evals}: \lambda_a.} A$$

$\therefore \| \Phi(U) \| \leq \| A^* \| \, \| \sum_a \lambda_a e_a e_a^* \| \, \|A\| \leq 1.$

For general $X \in L(\mathcal{X})$, $\frac{X}{\|X\|} = \sum_i q_i U_i$ (convex comb of unitaries)

$\therefore \| \Phi(\frac{X}{\|X\|}) \| = \| \Phi(\sum_i q_i U_i) \| \leq \sum_i q_i \| \Phi(U_i) \| \leq 1$

$\therefore \| \Phi(X) \| \leq \|X\|.$

<u>Thm 14.4</u> $\forall A \in \mathrm{Herm}(\mathcal{X} \otimes \mathcal{Y})$ s.t. $\|A\|_2 \leq 1$,

$\mathbb{1}_{\mathcal{X} \otimes \mathcal{Y}} - A \in \mathrm{Sep}(\mathcal{X} : \mathcal{Y})$.

NB: $\mathbb{1}_{\mathcal{X} \otimes \mathcal{Y}} - A \geq 0$

$\because \|A\| \leq \|A\|_2 \leq 1$

<u>Pf:</u> Assume $\mathcal{Y} = \mathbb{C}^{\Sigma}$, $A = \sum_{a,b \in \Sigma} A_{a,b} \otimes e_a e_b^*$

$\forall \Phi \in T(\mathcal{X}, \mathcal{Y})$ positive & unital,

$$(\Phi \otimes I)(A) = \sum_{a,b \in \Sigma} \Phi(A_{a,b}) \otimes e_a e_b^*$$

① $$\|(\Phi \otimes I)(A)\|^2 \underset{\text{Lem 14.2}}{\leq} \sum_{a,b \in \Sigma} \|\Phi(A_{a,b})\|^2$$

$$\underset{\text{Lem 14.3}}{\leq} \sum_{a,b \in \Sigma} \|A_{a,b}\|^2$$

$$\leq \sum_{a,b \in \Sigma} \|A_{a,b}\|_2^2 = \|A\|_2^2 \leq 1$$

$\|M\|^2$ = max singular value^2

$\|M\|_2^2 = \Sigma$ singular value^2

② $$\Phi \otimes I(\mathbb{1}_{\mathcal{X} \otimes \mathcal{Y}} - A) = \mathbb{1}_{\mathcal{Y} \otimes \mathcal{Y}} - \Phi \otimes I(A) \quad (\because \Phi \text{ unital})$$

$$\in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{Y}) \text{ by ①}.$$

By ② & Thm 14.1, $\mathbb{1}_{\mathcal{X} \otimes \mathcal{Y}} - A \in \mathrm{Sep}(\mathcal{X} : \mathcal{Y})$.