QIC 820 / CO781 / 486 / CS 867 Part 4 lecture 2

Last time: defined separable operators / states (mathematically)

They require no entanglement to prepare (operationally)

This lecture: equivalent concepts for operations

- Separable maps - cannot create / increase entanglement (Obs on P④)
- LOCC - does not require entanglement / Q communication

Operationally: ① expect LOCC $\subseteq$ SEP else we create entanglement from nothing

② LOCC is the class of interest

Mathematically: ① LOCC is unwieldy to study

② SEP is easier to study, and indeed LOCC $\subseteq$ SEP.

$\therefore$ SEP is often used to show no-go's for LOCC.

Roughly corr to Cp 15, additional discussion, Cp 16/13 (very brief statements),

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Cp 19, additional discussion.

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- First, we refine the Sep vs entangled distinction for operators.

Recall from Sec 2.4:

- $\mathrm{vec}(uv^*) = u \otimes \bar{v}$, $u \in \mathcal{X}$, $v \in \mathcal{Y}$
- $A = \sum_{i=1}^{r} s_i\, y_i\, x_i^*$ $\Leftrightarrow$ $\mathrm{vec}(A) = \sum_{i=1}^{r} s_i\, y_i \otimes \bar{x}_i = w$.

  Sing val decomp, rank $(A) = r$ — Schmidt decomp of $w \in \mathcal{Y} \otimes \mathcal{X}$, Schmidt rank of $w = r$

- $(A \otimes B)\,\mathrm{vec}(C) = \mathrm{vec}(A C B^T)$

<u>Def.</u> (min rank LN 2011, entanglement rank P.322 book)

$\mathrm{Ent}_r(\mathcal{X}:\mathcal{Y}) \subseteq \mathrm{Pos}(\mathcal{X}:\mathcal{Y})$

$R \in \mathrm{Ent}_r(\mathcal{X}:\mathcal{Y}) \Leftrightarrow R = \sum_{j=1}^{m} \mathrm{vec}(C_j) \cdot \mathrm{vec}(C_j)^*$

for some m, $\mathrm{rank}(C_j) \leq r$, $C_j \in L(\mathcal{Y}, \mathcal{X})$

$\Leftrightarrow R = \sum_{j=1}^{m} w_j \cdot w_j^*$

$w_j \in \mathcal{X} \otimes \mathcal{Y}$, Schmidt rank$(w_a) \leq r$

$\mathrm{Ent}_0(\mathcal{X}:\mathcal{Y}) = \{0\}$

$\mathrm{Ent}_1(\mathcal{X}:\mathcal{Y}) = \mathrm{Sep}(\mathcal{X}:\mathcal{Y})$

$\vdots$

$\mathrm{Ent}_k(\mathcal{X}:\mathcal{Y}) \subsetneq \mathrm{Ent}_{k+1}(\mathcal{X}:\mathcal{Y})$ for $k < n = \min(\dim(\mathcal{X}), \dim(\mathcal{Y}))$

$\mathrm{Ent}_n(\mathcal{X}:\mathcal{Y}) = \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$

<u>Obs:</u> Each $\mathrm{Ent}_r(\mathcal{X}:\mathcal{Y})$ is a <u>convex</u> cone, closed.

Obs: if $R \in \mathrm{Ent}_r(\mathcal{X}:\mathcal{Y})$, $A \in L(\mathcal{X},\mathcal{X}')$, $B \in L(\mathcal{Y},\mathcal{Y}')$

then $(A\otimes B)R(A\otimes B)^* \in \mathrm{Ent}_r(\mathcal{X}':\mathcal{Y}')$

Pf: If $\mathrm{rank}(C) \le r$

then $\mathrm{rank}(ACB^T) \le r$

Take $R \in \mathrm{Ent}_r(\mathcal{X}:\mathcal{Y})$, $R = \sum_{j=1}^{m} \mathrm{vec}(C_j)\cdot \mathrm{vec}(C_j)^*$, $\mathrm{rank}(C_j) \le r$

then
$$(A\otimes B)R(A\otimes B)^* = \sum_{j=1}^{m} \big(A\otimes B\, \mathrm{vec}(C_j)\big)\big(\mathrm{vec}(C_j)^*(A\otimes B)^*\big)$$
$$= \sum_{j=1}^{m} \mathrm{vec}(A C_j B^T)\cdot \big[\mathrm{vec}(A C_j B^T)\big]^*$$
$\mathrm{rank}(A C_j B^T) \le r$

$\in \mathrm{Ent}_r(\mathcal{X}:\mathcal{Y})$.

<u>Def (Separable maps)</u>:

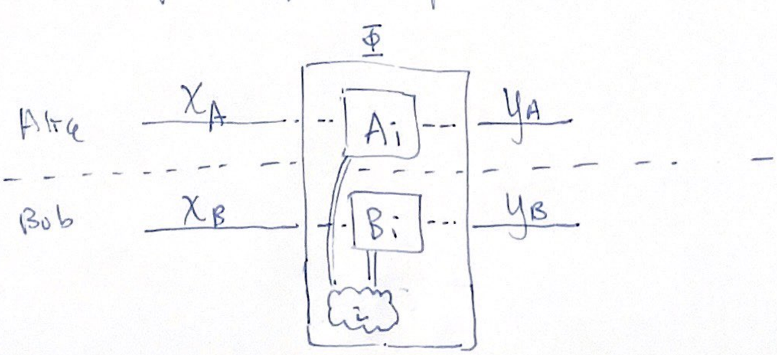
note (in: out partition)

Let $CP(\mathcal{X}_A\otimes\mathcal{X}_B, \mathcal{Y}_A\otimes\mathcal{Y}_B)$ be the set of completely positive maps from $\mathcal{X}_A\otimes\mathcal{X}_B$ to $\mathcal{Y}_A\otimes\mathcal{Y}_B$.

then $\Phi \in \mathrm{Sep}CP(\mathcal{X}_A \mathcal{Y}_A : \mathcal{X}_B \mathcal{Y}_B)$ (note: Alice : Bob partion)

if $\Phi(M) = \sum_{j=1}^{m} (A_i\otimes B_i)M(A_i\otimes B_i)^*$ for $A_i \in L(\mathcal{X}_A, \mathcal{Y}_A)$, $B_i \in L(\mathcal{X}_B, \mathcal{Y}_B)$

ie Kraus op are product operators.



NB: Def follows LN 2011 Sec 15.2 (15.1)

notation follows book

SepCP is defined differently in the book, will show equivalence later...

Define $\mathrm{SepC}(\mathcal{X}_A \mathcal{Y}_A : \mathcal{X}_B \mathcal{Y}_B) = \mathrm{SepCP}(\mathcal{X}_A \mathcal{Y}_A : \mathcal{X}_B \mathcal{Y}_B) \cap \mathrm{C}(\mathcal{X}_A \otimes \mathcal{X}_B, \mathcal{Y}_A \otimes \mathcal{Y}_B)$

$\uparrow$ Channels CP, TP $\qquad\qquad$ $\uparrow$ Channels CP, TP

NB: Since A_i, B_i not TP, Φ cannot be implemented by drawing i as shared randomness and Alice applies A_i, Bob applies B_i.

(If so, SEP = LOSR = LOCC very simple ....)

NB, most literature just write SEP for $\mathrm{SepC}(\mathcal{X}_A \mathcal{Y}_A : \mathcal{X}_B \mathcal{Y}_B)$.

<u>Thm 15.3</u> / <u>Thm 6.23</u> If $\Phi \in \mathrm{SepCP}(\mathcal{X}_A \mathcal{Y}_A : \mathcal{X}_B \mathcal{Y}_B)$, $R \in \mathrm{Ent}_r(\mathcal{X}_A : \mathcal{X}_B)$

then $\Phi(R) \in \mathrm{Ent}_r(\mathcal{Y}_A : \mathcal{Y}_B)$.

Pf: Let $\Phi(M) = \sum_{j=1}^{m} (A_j \otimes B_j) M (A_j \otimes B_j)^*$

from obs, each $\underbrace{(A_j \otimes B_j) R (A_j \otimes B_j)^*}$ $\in \mathrm{Ent}_r(\mathcal{Y}_A : \mathcal{Y}_B)$

same for the sum $\Phi(R) = \sum_j \ldots$ by convexity of $\mathrm{Ent}_r(\mathcal{Y}_A : \mathcal{Y}_B)$.

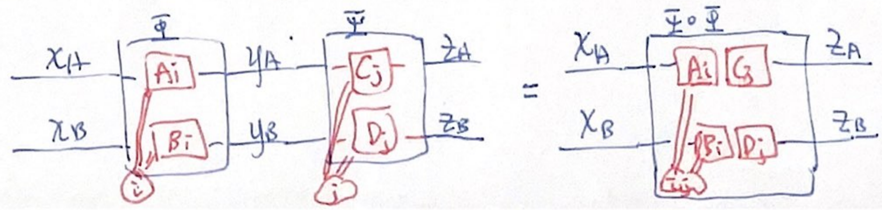
<u>Obs:</u> sep channels take sep op to sep op.

sep channels cannot increase entanglement rank.

<u>Prop 15.4 / Prop 6.19</u>: if $\Phi \in \mathrm{SepCP}(\mathcal{X}_A \mathcal{Y}_A : \mathcal{X}_B \mathcal{Y}_B)$

$\Psi \in \mathrm{SepCP}(\mathcal{Y}_A \mathcal{Z}_A : \mathcal{Y}_B \mathcal{Z}_B)$

then $\Psi \circ \Phi \in \mathrm{SepCP}(\mathcal{X}_A \mathcal{Z}_A : \mathcal{X}_B \mathcal{Z}_B)$



"Cute result" (3 notions of sep maps are equivalent, whew!)

Prop 15.1 / Prop 6.22: ① $\Phi \in \mathrm{SepCP}(\mathcal{X}_A \mathcal{Y}_A : \mathcal{X}_B \mathcal{Y}_B)$ (def by LN2011 (15.1))

$\Leftrightarrow$ ② $J(\Phi) \in \mathrm{Sep}(\mathcal{X}_A \mathcal{Y}_A : \mathcal{X}_B \mathcal{Y}_B)$ (def last lecture)

$\Leftrightarrow$ ③ $\Phi = \sum_{j=1}^{m} \Psi_j \otimes \Xi_j$, $\Psi_j \in CP(\mathcal{X}_A, \mathcal{Y}_A)$
$\Xi_j \in CP(\mathcal{X}_B, \mathcal{Y}_B)$

(def in book, 6.17)

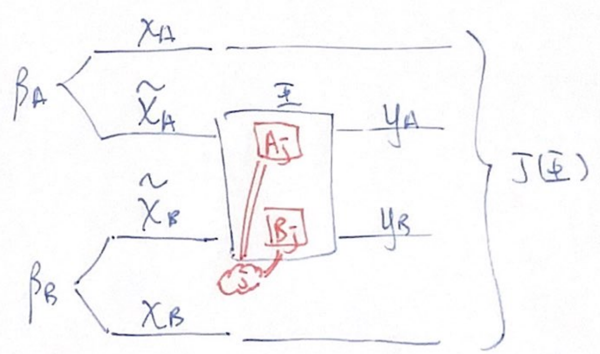
Pf: ① $\Rightarrow$ ②

$$\Phi(M) = \sum_{j=1}^{m} (A_j \otimes B_j) M (A_j \otimes B_j)^*$$

Let $\beta_A = \sum_{a=1}^{\dim(\mathcal{X}_A)} e_a \otimes e_a \in \tilde{\mathcal{X}}_A \otimes \mathcal{X}_A$

$\beta_B = \sum_{b=1}^{\dim \mathcal{X}_B} e_b \otimes e_b \in \tilde{\mathcal{X}}_B \otimes \mathcal{X}_B$.

$J(\Phi) = \Phi \otimes I(\beta_A \otimes \beta_B) =$



$$= \sum_j \underbrace{(A_j \otimes \mathbb{1}_{\mathcal{X}_A}) \beta_A \beta_A^* (A_j \otimes \mathbb{1}_{\mathcal{X}_A})^*}_{\substack{\text{in } \mathrm{Pos}(\mathcal{Y}_A \otimes \mathcal{X}_A) \\ \text{in fact } = J(\Psi_j) \\ \text{where } \Psi_j(M) = A_j M A_j^*}} \otimes \underbrace{(B_j \otimes \mathbb{1}_{\mathcal{X}_B}) \beta_B \beta_B^* (B_j \otimes \mathbb{1}_{\mathcal{X}_B})^*}_{\substack{\text{in } \mathrm{Pos}(\mathcal{Y}_B \otimes \mathcal{X}_B) \\ = J(\Xi_j) \\ \text{where } \Xi_j(M) = B_j M B_j^*}}$$

$\in \mathrm{Sep}(\mathcal{X}_A \mathcal{Y}_A : \mathcal{X}_B \mathcal{Y}_B)$

1 ⇒ 3

From above, $J(\Phi) = \sum_{j=1}^{m} J(\Psi_j) \otimes J(\Xi_j)$

$CP(\mathcal{X}_A \mathcal{X}_B, \mathcal{Y}_A \mathcal{Y}_B)$ $\quad CP(\mathcal{X}_A, \mathcal{Y}_A)$ $\quad CP(\mathcal{X}_B, \mathcal{Y}_B)$

in $L(\mathcal{Y}_A \mathcal{Y}_B \otimes \mathcal{X}_A \mathcal{X}_B)$ $\quad L(\mathcal{Y}_A \otimes \mathcal{X}_A)$ $\quad L(\mathcal{Y}_B \otimes \mathcal{X}_B)$

$\because$ Choi rep unique $\therefore$ $\Phi = \sum_{j=1}^{m} \Psi_j \otimes \Xi_j$

3 ⇒ 2 $\quad$ if $\Phi = \sum_{j=1}^{m} \Psi_j \otimes \Xi_j$

then $J(\Phi) = \sum_{j=1}^{m} J(\Psi_j) \otimes J(\Xi_j)$

$Pos(\mathcal{Y}_A \otimes \mathcal{X}_A)$ $\quad Pos(\mathcal{Y}_B \otimes \mathcal{X}_B)$ $\quad \because \Psi_j \in CP(\mathcal{X}_A, \mathcal{Y}_A)$

$\Xi_j \in CP(\mathcal{X}_B, \mathcal{Y}_B)$

$\therefore J(\Phi) \in Sep(\mathcal{Y}_A \otimes \mathcal{X}_A : \mathcal{Y}_B \otimes \mathcal{X}_B)$.

2 ⇒ 1 $\quad$ if $J(\Phi) \in Sep(\mathcal{Y}_A \otimes \mathcal{X}_A : \mathcal{Y}_B \otimes \mathcal{X}_B)$

then $J(\Phi) = \sum_{j=1}^{m} Q_j \otimes R_j$ $\quad$ where $Q_j \in Pos(\mathcal{Y}_A \otimes \mathcal{X}_A)$

$R_j \in Pos(\mathcal{Y}_B \otimes \mathcal{X}_B)$.

WLOG, $\operatorname{rank}(Q_j) = 1$, $\operatorname{rank}(R_j) = 1$

(by spectral decomp of Q_j, R_j and re-indexing j).

$\therefore Q_j = u_j u_j^*$ $\quad$ for $u_j \in \mathcal{Y}_A \otimes \mathcal{X}_A$

$= \operatorname{vec}(A_j) \cdot \operatorname{vec}(A_j)^*$ $\quad$ for $A_j \in L(\mathcal{X}_A, \mathcal{Y}_A)$

$= (A_j \otimes \mathbb{1}_{\mathcal{X}_A})\, \beta_A \beta_A^* \,(A_j \otimes \mathbb{1}_{\mathcal{X}_A})$:

Similarly $R_j = (B_j \otimes \mathbb{1}_{X_B})\, \beta_B \beta_B^* \,(B_j \otimes \mathbb{1}_{X_B})$.

$$\therefore\ J(\Phi) = \sum_{j=1}^{m} (A_j \otimes \mathbb{1}_{X_A})\, \beta_A \beta_A^* \,(A_j \otimes \mathbb{1}_{X_A})^* \otimes (B_j \otimes \mathbb{1}_{X_B})\, \beta_B \beta_B^* \,(B_j \otimes \mathbb{1}_{X_B})$$

$[\text{in } L(Y_A \otimes X_A \otimes Y_B \otimes X_A)]$

$$= \sum_{j=1}^{m} \left[(A_j \otimes B_j) \otimes \mathbb{1}_{X_A \otimes X_B}\right] (\beta_A \beta_B)(\beta_A \beta_B)^* \left[(A_j \otimes B_j) \otimes \mathbb{1}_{X_A \otimes X_B}\right]^*$$

$\text{in } L(Y_A \otimes Y_B \otimes X_A \otimes X_B)$

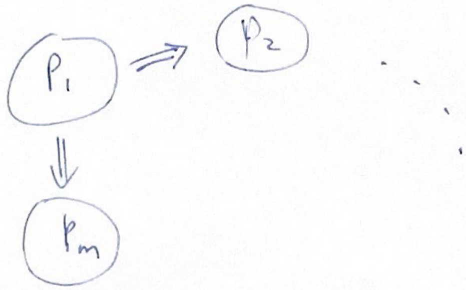
$\therefore$ Choi rep unique

$$\Phi(M) = A_j \otimes B_j \;\; M \;\; (A_j \otimes B_j)^*.$$

LOCC: local operations and classical communication

(as class of op that does not require Ent or Q channels).

Informally: m parties, each can perform any local operations (Q channels)



and they can send classical data to one another.

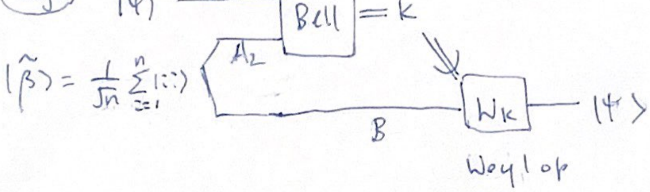
Results in a set of channels, each with m inputs & m outputs.

↑

called "LOCC"

These channels use no entanglement / Q comm but can take pure / mixed, separable / entangled inputs.

eg 1



Teleportation as LOCC channel

taking $|\psi\rangle_{A_1} |\tilde{\beta}\rangle_{A_2 B}$

to $\frac{1}{n^2} \sum_{k=1}^{n^2} |k\rangle\langle k|_{A_1 A_2} \otimes |k\rangle\langle k|_{B_1 B_2} \otimes |\psi\rangle_B$

eg 2 Entanglement dilution & concentration

$$\approx |\psi\rangle^{\otimes n} \rightleftarrows \approx \left[\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right]^{\otimes n S(\mathrm{tr}_B |\psi\rangle\langle\psi|)}$$

LOCC: size of system held by each party can change

- Each party can add $|0\rangle$ or $U|0\rangle$ locally.
- Each party can discard system with/without correlations with others.
- Classical comm from party i to party j can be modelled as:

$$\sum_{\substack{c \\ \text{(messages)}}} (\mathbb{1}_{X_1 \otimes \cdots \otimes X_m} \otimes \langle c|)\, \rho\, (\mathbb{1}_{X_1 \otimes \cdots \otimes X_m} \otimes |c\rangle) \otimes \underbrace{|c\rangle\langle c|}_{M_i} \otimes \underbrace{|c\rangle\langle c|}_{M_j}$$

ρ: state before CC: in $D(X_1 \otimes \cdots \otimes X_m \otimes M_i)$

X_k held by party k

M_i - - - - - i

M_i: party i retains a copy of the message

M_j: party j gains a copy of the message

- Obs: CC is a sep operation
- Obs: composing LO (sep) and CC (sep) is sep.

$\therefore$ LOCC $\subseteq$ SEP.

eg 3   Lo-Popescu 9707038

2 party, known pure state input $|\psi\rangle_{AB}$

$$\underset{\substack{\uparrow \\ \text{arbitrary} \\ \text{2-way CC}}}{LOCC} = \underset{\substack{\uparrow \\ \text{only 1 classical message from Alice to Bob}}}{LOCC_1} \underset{\neq}{\supset} LO$$

Lemma: if $|\psi\rangle_{AB} = \sum_k \sqrt{\lambda_k}\, |a_k\rangle |b_k\rangle$ Schmidt decomp

and Bob is to perform measurement with POVM $\{M_\ell\}$

ie producing the state:

$$|\psi'\rangle_{AB'B''} = \sum_k \sqrt{\lambda_k}\, |a_k\rangle_A \left(\sqrt{M_\ell}\, |b_k\rangle\right)_{B'} \otimes |\ell\rangle_{B''}$$

then $\exists$ POVM $\{M_\ell'\}$ for Alice s.t.

$$|\psi''\rangle_{A'A''B} = \sum_k \sqrt{\lambda_k} \left(\sqrt{M_\ell'}\, |a_k\rangle\right)_{A'} |\ell\rangle_{A''} \otimes |b_k\rangle$$

and $\forall \ell$, $\exists$ isometries U_ℓ, V_ℓ

s.t. $\sum_k \sqrt{\lambda_k}\, |a_k\rangle_A \otimes \left(\sqrt{M_\ell}\, |b_k\rangle\right)_{B'} = U_\ell \otimes V_\ell \left(\sum_k \sqrt{\lambda_k} \left(\sqrt{M_\ell'}\, |a_k\rangle\right)_{A'} \otimes |b_k\rangle_B \right)$

Pf: Express $\sqrt{M_\ell} = \sum_{ij} c_{ij}\, |b_i\rangle\langle b_j|$, let $\sqrt{M_\ell'} = \sum_{ij} c_{ij}\, |a_i\rangle\langle a_j|$.

$\therefore |u_\ell\rangle = \sum_k \sqrt{\lambda_k}\, |a_k\rangle_A \otimes \left(\sqrt{M_\ell}\, |b_k\rangle\right)_{B'} = \sum_k \sqrt{\lambda_k}\, |a_k\rangle_A \otimes \left(\sum_{ik} c_{ik} |b_i\rangle\right)_{B'}$

$|u_\ell'\rangle = \sum_k \sqrt{\lambda_k} \left(\sqrt{M_\ell'}\, |a_k\rangle\right)_{A'} \otimes |b_k\rangle_B = \sum_k \sqrt{\lambda_k} \left(\sum_{ik} c_{ik} |a_i\rangle\right)_{A'} \otimes |b_k\rangle_B$

if we transform $|a_i\rangle$ to $|b_i\rangle$ on A'

$|b_k\rangle$ to $|a_k\rangle$ on B

get $|u_\ell''\rangle = \sum_k \sqrt{\lambda_k} \left(\sum_{ik} c_{ik} |b_i\rangle\right)_{A'} \otimes |a_k\rangle_B$

$= (SWAP_{AB'})\, |u_\ell\rangle$

$\therefore$ $|u_l''\rangle$ and $|u_l\rangle$ have same schmidt coeffs

$\therefore$ $|u_l'\rangle$ and $|u_l\rangle$ - - - - - .

$\therefore$ $|u_l\rangle = U_l \otimes V_l\, |u_l'\rangle$ for some isometries U_l, V_l.

Cor: Alice can perform Bob's meas, obtain l

for each outcome l, she knows what $U_l \otimes V_l$ is needed

to obtain the "correct" $|u_l\rangle$ from her $|u_l'\rangle$.

Original 2-way protocol:	New 1-way protocol
Step 0 : $\lvert\psi\rangle_{AB}$	Step 0 : $\lvert\psi\rangle_{AB}$
Step 1 : Bob meas, gets outcome l sends l to Alice	Step 1 : Alice meas, gets outcome l applies U_l, imagines Bob applies V_l.
Step 2 : Alice meas, gets outcome c sends c to Bob	Step 2 : Alice meas, gets outcome c keeps c to herself.

Updated $|\tilde{\psi}\rangle_{AB}$ $\qquad$ $|\tilde{\psi}\rangle_{AB}$ if Bob applies V_l.

Steps 3-4 similar to step 1-2, apply same trick

- - - 5-6 - - - - - - - - .

Last step Alice measures, in the new 1-way protocol, she sends <u>ALL</u> meas outcomes to Bob so he knows l in step 1, applies U_l

- - - - 3, - - - .

- - - - 5, - - - .

Same final state!!

NB: Works for indefinite # steps, adaptive meas, prob final output etc.

eg 4 Thm 16.1 (Nielsen)

Let $x \in \mathcal{X}_A \otimes \mathcal{X}_B$, $y \in \mathcal{Y}_A \otimes \mathcal{Y}_B$ be arbitrary bipartite pure states.

Then $\exists\, \Phi \in \mathrm{LOCC}(\mathcal{X}_A \mathcal{Y}_A : \mathcal{X}_B \mathcal{Y}_B)$ s.t. $\Phi(xx^*) = yy^* \iff \mathrm{Tr}_{\mathcal{X}_B} xx^* \prec \mathrm{Tr}_{\mathcal{Y}_B} yy^*$.

Pf: Entire Ch 16. Reading Ex.

Def. For $C, D \in \mathrm{Herm}(\mathcal{X})$, say $D \prec C$,

if $\exists$ mixed unitary channel $\Phi \in \mathrm{C}(\mathcal{X})$ s.t. $D = \Phi(C)$

Def. for $u, v \in \mathbb{R}^{\Sigma}$, say $v \prec u$

if $\forall\ 1 \leq k < |\Sigma|$, $\displaystyle\sum_{j=1}^{k} v_j \leq \sum_{j=1}^{k} u_j$

and $\displaystyle\sum_{j=1}^{|\Sigma|} v_j = \sum_{j=1}^{|\Sigma|} u_j$

where we order the entries of u as $u_1 \geq u_2 \geq \dots \geq u_{\Sigma}$

" " " " v $v_1 \geq v_2 \geq \dots \geq v_{\Sigma}$

Thm 13.3. $C, D \in \mathrm{Herm}(\mathcal{X})$, $D \prec C \iff \lambda(D) \prec \lambda(C)$

Obs: more mixed $\prec$ less mixed

Thm 16.1: $\mathrm{Tr}_{\mathcal{X}_B} xx^* \prec \mathrm{Tr}_{\mathcal{Y}_B} yy^*$

more mixed / less mixed

$\therefore$ x more entangled / $\therefore$ y less entangled.

LOCC trf