QIC 820 / CO781 / 486 / CS 867 Part 4 lecture 4.

Highlights from LN 2011 lec 17, measures of entanglement.

Recall entanglement concentration & dilution:

$$|\psi\rangle^{\otimes n} \underset{\longleftarrow}{\xrightarrow{\text{LOCC}}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)^{\otimes n\, S(\mathrm{tr}_B |\psi\rangle\langle\psi|)}$$

Entanglement cost and distillable entanglement for a bipartite mixed state ρ are
$(E_C(\rho) = \inf E_1)$ $\quad$ $(E_d(\rho) = \sup E_2)$ motivated similarly:

$$\rho^{\otimes n} \xrightarrow{\text{LOCC}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)^{\otimes n E_1(\rho)}$$

$$\rho^{\otimes n} \longleftarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)^{\otimes n E_2(\rho)}$$

Thm 17.4 $\forall \rho \in D(X_A \otimes X_B)$, $E_d(\rho) \leq E_C(\rho)$.

Pf (read ex): relies on continuity & high fidelity of the prep & distillation tasks.
and LOCC NOT increase entanglement.

NB: The entanglement of formation,

$$E_f(\rho) = \min_{p_k, |\psi_k\rangle} \sum_k p_k \, S(\mathrm{tr}_B |\psi_k\rangle\langle\psi_k|)$$

thx to Carathéodory! $\quad$ s.t. $\sum_k p_k |\psi_k\rangle\langle\psi_k| = \rho$

is an upper bound for $E_C(\rho)$, due to the intuitive protocol:

1. Alice & Bob prepare $\approx n p_k$ copies of $|\psi_k\rangle$ for each k,

repeat n times:

2. prepare the state $\underbrace{\sum_k p_k |k\rangle\langle k|_{A'} \otimes |k\rangle\langle k|_{B'}}_{\text{implemented by CC}} \otimes \underbrace{|\psi_k\rangle\langle\psi_k|}_{\text{use the reservoir of prep states from (1)}}$

3. discard $A'B'$.

$E_f(\rho)$, proposed by BDSW96 (Bennett, DiVincenzo, Smolin, Wootters)
was thought to be $E_c(\rho)$ for a long while.

Hayden, (M) Horodecki, Terhal 0008134 showed:

$$E_c(\rho) = \lim_{k \to \infty} \frac{1}{k} E_f(\rho^{\otimes k})$$

but Hastings 2008 proved that

$$\exists\, \rho, \sigma \text{ s.t. } E_f(\rho_{AB} \otimes \sigma_{A'B'}) \lneq E_f(\rho_{AB}) + E_f(\sigma_{A'B'})$$

$\therefore$ $E_c(\rho) \leq E_f(\rho)$ in general.

$\lneq$ possible.

Highlights from lec18 :

- Recall from lec14 the transpose map $T(M) = M^t$ ← transpose
which is positive but not completely positive, and unital.

- Peres noted that all separable states are PPT (have positive partial transpose)

$$\rho = \sum_k p_k \, \sigma_k \otimes \eta_k \quad , \quad \sigma_k \in D(\mathcal{X}_A) \; , \; \eta_k \in D(\mathcal{X}_B)$$

then $I \otimes T(\rho) = \sum_k p_k \, \sigma_k \otimes \eta_k^T$ still in $D(\mathcal{X}_A \otimes \mathcal{X}_B)$.

$\therefore \; I \otimes T(\rho) \geq 0$

- Also, if $I \otimes T(\rho) \notin \mathrm{Pos}(\mathcal{X}_A \otimes \mathcal{X}_B)$ for $\rho \in D(\mathcal{X}_A \otimes \mathcal{X}_B)$
by Horodecki (M, P., R) criterion, $\rho \notin \mathrm{Sep}\,D(\mathcal{X}_A : \mathcal{X}_B)$

Equivalent . . . .

Converse: if $I \otimes T(\rho) \in \mathrm{Pos}(\mathcal{X}_A \otimes \mathcal{X}_B)$
is $\rho \in \mathrm{Sep}\,D(\mathcal{X}_A : \mathcal{X}_B)$?

Not necessarily !!

States PPT but not separable are called PPT <u>bound entangled</u> states.

$E_d(\rho) = 0$ not Sep

(1) Will see an example

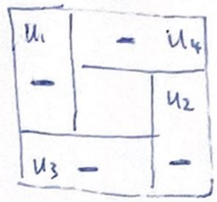
(2) Will see sep operations are "PPT preserving"

(3) Will see PPT states are not distillable.

Unextendible product "basis" and bound entanglement (9808030):

Let $u_1 = |0\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

4 comes from the nonlocality w/o ent ensemble.



$u_2 = |2\rangle \otimes \left(\frac{|1\rangle - |2\rangle}{\sqrt{2}}\right)$

$u_3 = \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \otimes |2\rangle$

$u_4 = \left(\frac{|1\rangle - |2\rangle}{\sqrt{2}}\right) \otimes |0\rangle$

$u_5 = \left(\frac{|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}}\right) \otimes \left(\frac{|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}}\right)$

- O.n set in $\mathbb{C}^3 \otimes \mathbb{C}^3$.
- Claim: if $\langle u, u_i\rangle = 0$ for $i = 1 \ldots 5$ then $u \neq$ product vector.

Pf: by contradiction.

let $u_i = x_i \otimes y_i$, $u = x \otimes y$

if $\underbrace{\langle u, u_i\rangle = 0 \ \forall i}$, then $\langle x, x_i\rangle = 0$ for at least 3 values of i

ie $\forall i$ either $\langle x, x_i\rangle = 0$ or $\langle y, y_i\rangle = 0$

But every set of 3 x_i's span $\mathbb{C}^3$ $\therefore$ $x = 0$ contradiction.

Now let $\rho_{AB} = \dfrac{I \otimes I - \sum_{i=1}^{5} u_i u_i^*}{4}$

$\forall i, \ \operatorname{tr} \rho_{AB} u_i u_i^* = 0$

If $\rho_{AB} \in \operatorname{SepD}(\mathbb{C}^3 : \mathbb{C}^3)$, $\rho_{AB} = \sum_k p_k (x_k \otimes y_k)(x_k \otimes y_k)^*$

$\Rightarrow \forall i \ \langle x_k \otimes y_k, u_i\rangle = 0$ contradict claim.

$\therefore \rho_{AB} \notin \operatorname{SepD}(\mathbb{C}^3 : \mathbb{C}^3)$.

Thm 18.1: If ρ_{AB} PPT, $\Phi \in \mathrm{Sep}((X_A, Y_A : X_B, Y_B)$

then $\Phi(\rho_{AB})$ PPT

ie Any SEP channel preserves PPT-ness.

Pf: Note PPT operators form a convex set.

$\therefore$ suffices to show ρ_{AB} PPT $\Rightarrow$ $(A\otimes B)\,\rho\,(A\otimes B)^*$ PPT

First, $(T\otimes I)(\mathbb{1}\otimes B)\,\rho\,(\mathbb{1}\otimes B^*)$

$$= (\mathbb{1}\otimes B)\underbrace{(T\otimes I)(\rho)}_{\geq 0}(\mathbb{1}\otimes B^*) \geq 0$$

Second $T\otimes T\left((\mathbb{1}\otimes B)\; T\otimes I(\rho)\;(\mathbb{1}\otimes B^*)\right) \geq 0$. from above

$\therefore$ $I\otimes T\left((\mathbb{1}\otimes B)\,\rho\,(\mathbb{1}\otimes B^*)\right) \geq 0$

Finally $(A\otimes \mathbb{1})\left((I\otimes T)(\mathbb{1}\otimes B)\,\rho\,(\mathbb{1}\otimes B^*)\right)(A^*\otimes \mathbb{1}) \geq 0$

from above

$$\| $$

$$I\otimes T\left[(A\otimes B)\,\rho\,(A\otimes B)^*\right]$$

NB: What channels preserve PPT-ness?

This class is called PPT-channels !!

Will see in HS $\mathrm{SEP} \subsetneq \mathrm{PPT}$, $J(\Phi)$ PPT if Φ PPT !

Thm 18.3 if ρ_{AB} PPT, then $E_d(\rho) = 0$.

Pf: $\rho^{\otimes n}$ PPT (on the partition $A_1 A_2 \cdots A_n : B_1 B_2 \ldots B_n$)

for any distillation protocol Φ in LOCC $\subseteq$ SEP.

$\Phi(\rho^{\otimes n})$ PPT

if $\Phi(\rho^{\otimes n})$ outputs one ebit with high fidelity,

that ebit is NOT PPT. (Part 4 sec 1).

Remarks:

- If ρ NPT, ie $I \otimes T(\rho) \ngeq 0$, is $E_d(\rho) > 0$?

  Open problem for 25+ years!!

  Conjecture: $\exists$ NPT BE states.

- PPT channels are useful in analysis.

  The condition: CP, TP, PPT all semidefinite!! So SDP optimizing

  PPT channels are heavily used.

  It limits what LOCC / SEP can achieve.

  (see AS for Q data hiding.)

- Rains used PPT distillation protocols to upper bound E_d.

- BDSW 96 used E_d (1-way LOCC), E_d (2-way LOCC)

  to study capacity of q. channels to transmit q. data!!

- PPT channels have no Q capacity, PPT states have no E_d.

  But they can have positive key or private classical capacity HHHO 03

  KMP oppenheim