# Fall 2016 QIC 890 / CO 781 Assignment 1

Due Sept 30 or Oct 03, 2016 (in my QNC mailbox)

**Question 1. Impossibility to send quantum states via a classical channel (short proof)**

Prove that, $\forall n \in \mathbb{Z}^+$, $n$ cbits $\not\geq$ 1 qbit.

Hint: Show that if Alice can encode an arbitrary one-qubit state into classical bits, and Bob can retrieve $|\psi\rangle$, then, there is a method to clone an arbitrary qubit state. You can assume the no-cloning theorem in quantum mechanics.

**Question 2. Impossibility to send quantum states via a classical channel (long proof)**

Prove that, $\forall n \in \mathbb{Z}^+$, $n$ cbits $\not\geq$ 1 qbit, by showing that $\forall n \in \mathbb{Z}^+$, $n$ cbits $\not\geq$ 1 ebit.

Hint: you can use the following line of argument.

A separable state on two systems $A$ and $B$ is represented by a density matrix of the form

$$\rho_{AB} = \sum_i p_i \, \alpha_A^{(i)} \otimes \beta_B^{(i)}$$

where $\alpha_A^{(i)}$ and $\beta_B^{(i)}$ are density matrices on systems $A$ and $B$ respectively, $p_i \geq 0$ and $\sum_i p_i = 1$.

Throughout this question, Alice can have multiple quantum systems in her laboratory collectively called $A$. Similarly for Bob and his combined system $B$. We allow the local systems $A$, $B$ to expand or shrink.

An operation $\mathcal{E}$ is said to be an LOCC operation if it is a composition of local operations (TCP maps acting only on $A$ or only on $B$) and classical communication in either direction.

(a) Show that any instrument on $A$ maps any separable state to another separable state.

An instrument is a quantum operation of the form $\mathcal{F}(\sigma) = \sum_k \mathcal{G}_k(\sigma) \otimes |k\rangle\langle k|$ where $\sum_k \mathcal{G}_k$ is a TCP map and $\{|k\rangle\}$ is a basis on some additional output system. (Both output systems belong to Alice here.) An instrument is sometimes called a generalized measurement or a non-demolition measurement. Note a TCP map is a special instrument, and an instrument is a TCP map.

(b) Show that classical communication in either direction maps any separable state to another separable state. (You may use the fact that an instrument is the most general operation that extracts classical data from a quantum state.)

(c) Show that any LOCC operation maps any separable state to another separable state.

(d) Show that in general the partial transpose of a separable state is another separable state.

(e) Show that the partial transpose of an ebit is not a state.

(f) Show that $\forall n \in \mathbb{Z}^+$, $n$ cbits $\not\geq$ 1 ebit.

**Question 3. Superdense coding of quantum states**

Let $A$ and $B$ be quantum systems of dimensions $d_A$ and $d_B$ respectively, for $d_A \geq d_B \geq 3$. A state $|\Psi\rangle \in A \otimes B$ is maximally entangled if $\mathrm{tr}_A |\Psi\rangle\langle\Psi| = \frac{I}{d_B}$.

You can assume the following fact from quant-ph/0407049:

Let $0 < \alpha < \log d_B$, $\beta = \frac{1}{\ln 2}\frac{d_B}{d_A}$, and $\Gamma$ be an absolute constant which may be chosen to be $1/1753$. There exists a subspace $S \subset A \otimes B$ of dimension

$$s = \left\lfloor d_A d_B \frac{\Gamma \alpha^{2.5}}{(\log d_B)^{2.5}} \right\rfloor$$

with the property that for every state $|\psi\rangle \in S$, there exists a maximally entangled state $|\Psi\rangle \in A \otimes B$ such that

$$\| |\psi\rangle\langle\psi| - |\Psi\rangle\langle\Psi| \|_1 \leq \sqrt[4]{16(\alpha + \beta)}.$$

Using the above fact, show that for any $d^2$-dimensional pure state $|\phi\rangle$ of Alice's choice, she can prepare $|\tilde{\phi}\rangle$ in Bob's laboratory given $\log d$ ebits and $\log d + 2.5 \log\log d - \log(\Gamma\alpha^{2.5})$ qbits, where

$$\| |\phi\rangle\langle\phi| - |\tilde{\phi}\rangle\langle\tilde{\phi}| \|_1 \leq 2\sqrt[4]{2\alpha}.$$

You can also use without proof that for any two maximally entangled states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ in $A \otimes B$, there exists a unitary $U \in \mathbb{U}(A)$ such that $|\Psi_2\rangle = (U \otimes I)|\Psi_1\rangle$