Fall 2016 QIC 890 / CO 781 Assignment 2

Due Oct 14, 2016

Question 1. Data compression

Consider a biased coin X, with sample space $\Omega = \{0, 1\}$ and probability distribution $p(0) = 0.8$, $p(1) = 0.2$.

(a) What is $H(X)$?

(b) Suppose 1000 iid draws of X are made, and the result be recorded in an 800-bit USB drive. Derive an upper bound of the decoding error.

Note: for (b), you can use Chebyshev's inequality as in the lecture, or other well known large deviation bounds if you wish (if the latter, give a precise statement of the bound involved).

Question 2. Entanglement concentration

Suppose Alice and Bob share $|\psi\rangle^{\otimes n}$, that is, n copies of the state

$$|\psi\rangle = \sqrt{a}\,|00\rangle + \sqrt{1-a}\,|11\rangle$$

where $a \in [0, 1]$, and the first qubit belongs to Alice, and the second to Bob. Denote Alice's n-qubit system by $A = A_1 \otimes A_2 \otimes \cdots \otimes A_n$, Bob's n-qubit system $B = B_1 \otimes \cdots \otimes B_n$.

Both Alice and Bob have the same reduced state $\rho^{\otimes n}$ where $\rho = a|0\rangle\langle 0| + (1-a)|1\rangle\langle 1|$.

Let $H(a) = -a \log a - (1 - a) \log(1 - a)$ (the binary entropy function) which is also $S(\rho)$ here. (We use capitalized H here because lower case h labels something else later.)

The goal is to show that for large n, *approximately* $nH(a)$ ebits can be obtained with local operations and no communication.

(a) The problem is reminiscent to Schumacher compression, but you will go through some analysis showing that there is insufficient accuracy.

Let $\epsilon > 0$, $\delta = H(a) - R$, n_0 and $T_{n,\delta}$ be as defined in class with respect to the random variable X with $p(0) = a$, $p(1) = 1 - a$. Let $S = \text{span}\{|x^n\rangle, x^n \in T_{n,\delta}\}$ and Π_S the projector onto S.

Suppose each of Alice and Bob applies the measurement with POVM $\{M_0 = \Pi_S, M_1 = I - \Pi_S\}$ on systems A and B respectively.

(i) Show that Alice and Bob always obtain the same outcome. Show that the joint event with both outcomes being "0" occurs with probability at least $1 - \epsilon$.

(ii) Write down the postmeasurement state $|\tilde{\Phi}\rangle$ if both outcomes are "0".

(iii) Define

$$|\Phi\rangle = \frac{1}{\sqrt{|T_{n,\delta}|}} \sum_{x^n \in T_{n,\delta}} |x^n\rangle|x^n\rangle.$$

Note that $|\Phi\rangle$ is exactly maximally entangled, and can be transformed into $\log |T_{n,\delta}| \geq n(H(a) - \delta)$ ebits if Alice and Bob apply some appropriate local unitaries.

Give upper and lower bounds for $|\langle\Phi|\tilde{\Phi}\rangle|$ based on what you learnt in class concerning typicality. Note how the lower bound behaves with large n.

(b) A better method:

For an n-bit string x^n, denote the hamming weight by $h(x^n)$, which is the number of 1's in x^n.

For $k \in \{1, \cdots, n\}$, let $S_k = \text{span}\{|x^n\rangle : h(x^n) = k\}$, and Π_k be the projector onto S_k.

Define a new measurement with POVM $\{\Pi_0, \Pi_1, \cdots, \Pi_n\}$ (and denote the corresponding outcome by the subscript).

(i) Show that Alice and Bob always get the same outcome. What is the probability they both get k?

(ii) Write down the *normalized* state $|\Phi_k\rangle$ conditioned on both Alice and Bob obtaining outcome k. Note that it is maximally entangled.

(iii) Show that the *expected* number of ebits Alice and Bob can obtain is $H(X^n|K)$ where K is the random variable associated with Alice's measurement outcome.

(iv) Show that $H(X^n|K) \geq nH(a) - \log n$.

(v) Why is communication not needed?

NB. The expression for the *expected* number of ebits is $\sum_{k=0}^{n} \binom{n}{k} a^{n-k}(1-a)^k \log \binom{n}{k}$. It is not so easy to lower bound directly.

NB To simplify the question, we ignore the possibility that the postmeasurement maximally entangled states need not have dimension which is a power of 2. This costs only a slight reduction in the yield.

NB The binary X can be generalized, and the final answer has $H(X)$ in place of $H(a)$, log(number of type classes) instead of $\log(n+1)$.

Question 3. Reading exercise of p3-p8 of notes for lecture 8

Question 4. Proof of JAEP in lecture 8 [Practice question, do not turn in.]

Question 5. Strong typicality [Practice question, do not turn in.]
Let X has sample space $\Omega = \{1, 2, \cdots, m\}$ and distribution $p(a)$.
Given a block of n symbols $x^n = x_1 x_2 \cdots x_n$, $\forall a \in \Omega$, let $N(a|x^n) =$ number of times a appears in x^n.
In other words, $\frac{1}{n} N(a|x^n)$ is the empirical distribution of x^n.
For example, $m = 3$, $n = 10$, $x^n = 2213123112$, $N(1|x^n) = N(2|x^n) = 4$, $N(3|x^n) = 2$.
We say that x^n is η-strongly typical if

$$\left|\frac{1}{n} N(a|x^n) - p(a)\right| \leq \eta \text{ whenever } p(a) > 0$$

$$N(a|x^n) = 0 \text{ otherwise.}$$

Show that if x^n is η-strongly typical, it is ηc-typical where $c = -\sum_{a:p(a)>0} \log(p(a))$.