Fall 2016 QIC 890 / CO 781 Assignment 3

Due Nov 10 (preferred) – 15 (latest), 2016

Question 1. Accessible information

Consider the ensemble $\mathcal{E}_2 = \{p(x), \rho_x\}_{x=0}^{2}$,
where $p(x) = 1/3$ for $x = 0, 1, 2$, $\rho_x = |\psi_x\rangle\langle\psi_x|^{\otimes 2}$,
$|\psi_0\rangle = |0\rangle$, $|\psi_1\rangle = \cos(\pi/3)|0\rangle + \sin(\pi/3)|1\rangle$, $|\psi_2\rangle = \cos(2\pi/3)|0\rangle + \sin(2\pi/3)|1\rangle$,

(Note: $\mathcal{E}_2$ is as defined in class.)

Calculate the mutual information between X and the measurement outcome Y for the two measurements $\mathcal{M}_2$ and $\mathcal{M}_3$ as defined in p13-14 of notes for lecture 11, Oct 18.

(Hint: the answers are ≈ 1.2304 and 1.3695 respectively.)

Question 2. Some simple exercises on Q boxes

Throughout this question, states are in $\mathbb{C}^2$, and when you are asked to calculate the capacities, **derive the optimal distribution as well**. Hint: $C(Q) = 1$ and ≈ 0.558 for (a) and (b).

(a) Let I, σ_x, σ_y, σ_z denote the Pauli matrices.

$|\psi_1\rangle\langle\psi_1| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(\sigma_x + \sigma_y + \sigma_z))$

$|\psi_2\rangle\langle\psi_2| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(\sigma_x - \sigma_y - \sigma_z))$

$|\psi_3\rangle\langle\psi_3| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(-\sigma_x + \sigma_y - \sigma_z))$

$|\psi_4\rangle\langle\psi_4| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(-\sigma_x - \sigma_y + \sigma_z))$

Note that $|\langle\psi_i|\psi_j\rangle|$ is constant for $i \neq j$, and the Bloch vectors of these states form the vertices of a tetrahedron.

(i) What is the pretty good measurement corresponding to these states?

(ii) What is the classical capacity of a Q-box capable of emitting $|\psi_i\rangle$ ($i = 1, 2, 3, 4$)?

(b) Consider the states $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = \frac{1}{2}(\frac{I}{2} + |1\rangle\langle 1|)$.

(i) What is the pretty good measurement corresponding to these states?

(ii) What is the classical capacity of a Q-box capable of emitting ρ_0 and ρ_1?

Question 3. The optimal ensemble for $\chi(\mathcal{N})$.

Recall $\chi(\mathcal{N}) = \max_{p_x, \rho_x} S(\mathcal{N}(\sum_x p_x \rho_x)) - \sum_x p_x S(\mathcal{N}(\rho_x))$.

Note that

$$\chi(\mathcal{N}) = \max_{\rho} \left[S(\mathcal{N}(\rho)) - \min_{\substack{p_x, \rho_x: \\ \sum_x p_x \rho_x = \rho}} \sum_x p_x S(\mathcal{N}(\rho_x)) \right]$$

Assume that for any ρ, the minimization of the second term above can be attained by some ensemble $\{p_x, \rho_x\}$ where there is no bound on the number of states in the ensemble.

(a) Show that there is some ensemble $\{q_t, |\psi_t\rangle\langle\psi_t|\}$ achieving the same minimum.

The Caratheodory's theorem says, for any set S in $\mathbb{R}^n$, if $a \in \mathrm{conv}(S)$, then, a is a convex combination of some $n+1$ elements of S.

(b) Use Caratheodory's theorem to show that there is a distribution r_t such that $\{r_t, |\psi_t\rangle\langle\psi_t|\}$ attains the same minimum, but $r_t > 0$ for only d^2 values of r.