Two more operational meanings for $S(\rho)$:

① Consider the following task.

Referee prepares $|\psi\rangle_{RA}^{\otimes n}$,

gives $A_1 A_2 \ldots A_n$ to Alice

and she should transmit $A_1 \ldots A_n$

to Bob preserving correlation ← this time, correlation is entanglement

with $R_1 \ldots R_n$.

* Note: Alice & Bob know $|\psi\rangle$ here.

Let $\rho = \mathrm{Tr}_R\, |\psi\rangle\langle\psi|_{RA}$.

Once again, transmitting the δ-typical space of $\rho^{\otimes n}$ achieves the goal with rate $R \approx S(\rho)$. See Preskill last part of Sec 10.3

The resulting state in this case is (easiest to think in Schmidt basis).

$$\rho_{out} = I \otimes \Pi_S\, |\psi\rangle\langle\psi|^{\otimes n}_{R^n A^n}\, I \otimes \Pi_S + \underbrace{\mathrm{tr}\left[I \otimes (I - \Pi_S)\, |\psi\rangle\langle\psi|^{\otimes n} \right]}_{\substack{\| \\ \mathrm{tr}(\Pi_S \rho^{\otimes n}) \le \varepsilon}} \cdot ERR$$

$$\| \rho_{out} - |\psi\rangle\langle\psi|^{\otimes n}_{R^n A^n} \|_1$$

Δ ineq

$$\le \| I \otimes \Pi_S\, |\psi\rangle\langle\psi|^{\otimes n}_{R^n A^n}\, I \otimes \Pi_S - |\psi\rangle\langle\psi|^{\otimes n}_{R^n A^n} \|_1 + \varepsilon,$$

minor detail notes →

$$\le 2\sqrt{1 - \left(\langle\psi|^{\otimes n}_{R^n A^n}\, I \otimes \Pi_S\, |\psi\rangle^{\otimes n}_{R^n A^n} \right)^2} + \varepsilon$$

$$\le 2\sqrt{2}\, \sqrt{1 - \langle\psi|^{\otimes n}_{R^n A^n}\, I \otimes \Pi_S\, |\psi\rangle^{\otimes n}_{R^n A^n}} + \varepsilon$$

Same

$$\le 2\sqrt{2}\, \sqrt{\varepsilon} + \varepsilon.$$

Description of $|\psi\rangle$ known to Alice & Bob throughout

② <u>Entanglement concentration & dilution</u> Bennett-Bernstein-Popescu-Schumacher 9511030

Two related questions:

ⓐ How many ebits are required to prepare $|\psi\rangle_{AB}^{\otimes n}$ with high fidelity?

ⓑ How many ebits can be extracted from $|\psi\rangle_{AB}^{\otimes n}$ with high fidelity?

We allow arbitrary back & forth classical communication between A & B and local operations (LOCC) but nothing else.

ignoring $o(n)$ terms

Answer: $\simeq n S(\mathrm{tr}_A |\psi\rangle\langle\psi|)$ for both questions.

ⓘ The fact there <u>is</u> an answer for either question is a surprise. For pure states shared among 3 or more parties, the questions cannot even be well formulated (no equivalence of the ebit).

ⓘⓘ For bipartite mixed states, answer for ⓐ $\underset{\neq}{>}$ answer for ⓑ in general. So there is no unique quantification of entanglement.

ⓘⓘⓘ For bipartite pure states, because of the surprisingly nice answers to ⓐ & ⓑ, $S(\mathrm{tr}_A |\psi\rangle\langle\psi|)$ is a useful operational way to quantify entanglement in $|\psi\rangle$.

ⓘⓥ Method for ⓐ is called entanglement dilution
— — ⓑ — — — — concentration.

ⓥ Even more surprising: conc requires NO classical communication. Dilute requires $o(n)$ cbits (and $\Omega(\sqrt{n})$ suffices).
Harrow-Lo 02, Hayden-Winter 02, first asked by Lo-Popescu 9902045.

For detail, see Preskill 10.4.

(a) For dilution the protocol to distribute $|\psi\rangle_{RA}^{\otimes n}$ using $\approx n S(\mathrm{tr}_A |\psi\rangle\langle\psi|)$ qbits can be converted to one using $\approx n S(\mathrm{tr}_A |\psi\rangle\langle\psi|)$ ebits $+ 2nS(\mathrm{tr}_A |\psi\rangle\langle\psi|)$ cbits due to teleportation. This already proves the claim, though unnecessarily too many cbits are used. $\boxed{\text{See diagram end of pdf file.}}$

(b) For concentration: See A2.

Idea: Alice & Bob can apply local unitaries to turn $|\psi\rangle_{AB}$ into the Schmidt form

$$|\psi\rangle_{AB} = \sum_x \sqrt{p(x)}\, |x\rangle|x\rangle.$$

$$|\psi\rangle_{AB}^{\otimes n} = \sum_{x^n} \sqrt{p(x^n)}\, |x^n\rangle|x^n\rangle.$$

* The tempting method is for both Alice & Bob to meas their n system w/ POVM $\{\Pi_S, I-\Pi_S\}$.

They get, with high prob $\propto \sum_{x^n\in T_{n,\delta}} \sqrt{p(x^n)}\, |x^n\rangle|x^n\rangle$.

"roughly" equal.

BUT it is not easy to lower bound the fidelity of the above state & the maximally entangled state

$$\sum_{x^n\in T_{n,\delta}} |x^n\rangle|x^n\rangle \frac{1}{\sqrt{|T_{n,\delta}|}}.$$

* A good method is for both Alice & Bob to make a much <u>finer</u> measurement (resulting state closer to MES but has fewer Schmidt term)

<u>Def.</u>: For X with sample space $\Omega = \{1, 2, \ldots, m\}$

$x_1, x_2 \ldots x_n$ is in the <u>type class</u> $(t_1, t_2, \ldots, t_m)$

where t_k = # x_i's equal to k.

eg. $m = 4$, $n = 10$

$$\begin{array}{ccc} & & \text{2 1's} \quad \text{3 2's} \quad \text{3 3's} \quad \text{2 4's} \\ 1\,4\,2\,3\,3\,2\,1\,2\,4\,3 & \text{is in the type class} & (2,\ 3,\ 3,\ 2). \end{array}$$

eg. n coin tosses, any n-bit outcome with

1's in the n bit string $\longrightarrow$ Hamming weight k is in the type class $(n-k, k)$.

All x^n's in the same type class are <u>exactly</u> equiprobable.

To concentrate entanglement, Alice & Bob independently measure the type class. They always get the same outcome.

The postmeasurement state is exactly maximally entangled but with a probabilistic amount of entanglement.

You will show in A2 the expected amount is

$$S(\mathrm{tr}_A |\psi\rangle\langle\psi|) \cdot n - o(n)$$

for $|\psi\rangle \in \mathbb{C}^{2 \otimes 2}$ but idea generalizes to arbitrary $|\psi\rangle$.

as long as # type classes is small enough (as long as n large enough).

The dilution protocol discussed in class:

Given: $n(S(\rho)+\delta)$ ebits between Alice & Bob, LOCC.

Want: $\left(|\psi\rangle_{AB}\right)^{\otimes n}$

* Both Alice & Bob know what $|\psi\rangle$ is