Last time

HSW Thm

$$C(N) = \sup_r \chi^{(r)}(N)$$

where $\chi^{(r)}(N) = \frac{1}{r}\chi(N^{\otimes r})$

$$\chi(N) = \max_{\{p_x, \rho_x\}} \chi(\{p_x, N(\rho_x)\})$$

Consequences:

① $C(N) = \sup_r \chi^{(r)}(N) \geq \chi^{(r)}(N) \geq \chi^{(1)}(N)$

② $C(N) = 0 \iff \chi^{(1)}(N) = 0 \iff N$ "rubbish channel"

③ $C(N)$ cts in N for the metric derived from the diamond norm.

⑤ How to find $\chi^{(1)}(N)$?

Def: In the expression $\chi(\mathcal{N}) := \max_{\{p_x, \rho_x\}} \chi(\{p_x, \mathcal{N}(\rho_x)\})$

$\{p_x, \rho_x\}$ is called the "optimal ensemble" for $\mathcal{N}$ if it achieves the max.

Properties of the optimal ensemble:

Uhlmann 9701014, Schumacher & Westmoreland 9912122

① ρ_x's can be chosen pure

② d^2 pure states are sufficient, where d = output dim of $\mathcal{N}$

Pf: A3 (similar to I_{acc})

eg 1 $\mathcal{N}_p$ = d-dim depolarizing channel

$$\mathcal{N}_p(\rho) = (1-p)\,\rho + p\,\frac{I}{d}$$

$$\chi^{(1)}(\mathcal{N}_p) = \max_{p_x, |\psi_x\rangle} S\Big(\sum_x p_x \mathcal{N}(|\psi_x\rangle\langle\psi_x|)\Big) - \sum_x p_x S(\mathcal{N}(|\psi_x\rangle\langle\psi_x|))$$

For the 2nd term:

$$\mathcal{N}(|\psi_x\rangle\langle\psi_x|) = (1-p)\,|\psi_x\rangle\langle\psi_x| + p\,\frac{I}{d}.$$

Spectrum $= \Big(1-p+\frac{p}{d}, \underbrace{\frac{p}{d}, \ldots \frac{p}{d}}_{d-1 \text{ times}}\Big)$ independent of $|\psi\rangle$.

$$S(\mathcal{N}(|\psi_x\rangle\langle\psi_x|)) = -\Big(1-p+\frac{p}{d}\Big)\log\Big(1-p+\frac{p}{d}\Big) - \frac{d-1}{d}\,p\log\frac{p}{d} =: s$$

$$\therefore \chi^{(1)}(\mathcal{N}_p) = \underbrace{\max_{p_x, |\psi_x\rangle} S\Big(\sum_x p_x \mathcal{N}(|\psi_x\rangle\langle\psi_x|)\Big)}_{\text{attained when } \sum_x p_x \mathcal{N}(|\psi_x\rangle\langle\psi_x|) = \frac{I}{d}} - s$$

when $p_x = \frac{1}{d}$, $|\psi_x\rangle = |x\rangle$.

$$= \log d - s.$$

Cor $d=2$, $\chi^{(1)}(\mathcal{N}_p) = \underbrace{1 - h\big(\tfrac{p}{2}\big)}_{=}$ (will see this is $C(\mathcal{N}_p)$)

Capacity of classical BSC with flipping prob $\frac{p}{2}$.

What is the code for achieving this? (d-dim case)

Draw $\approx 2^{n\chi}$ code words from all possible n-tuples of symbols in $\{1, 2, \ldots, d\}$ and restrict to strongly-typical sequences.

Due to overlaps of the output pts, expect Γ entangling & so is the decoding measurements.

In general, if ① N unital $\quad (N(I) = I)$

② each $|\psi_i\rangle$ minimizes $S(N(|\psi_i\rangle\langle\psi_i|)) =: S$

③ $\frac{I}{d} \in \mathrm{conv}(|\psi_i\rangle)$

then $\{p_i, |\psi_i\rangle\}$ optimal ensemble and $\chi^{(1)}(N) = \log d - S$

where $\sum_i p_i |\psi_i\rangle\langle\psi_i| = \frac{I}{d}$.

Ex: find $\chi^{(1)}(N)$ for $N(\rho) = 0.8\,\rho + 0.15\, X\rho X + 0.05\, Z\rho Z$

Ex: find $\chi^{(1)}(E_p)$ for $\underset{\substack{\uparrow \\ \text{d-dim} \\ \text{erasure} \\ \text{channel}}}{E_p(\rho)} = (1-p)\,\rho + \underset{\substack{\uparrow \\ \text{erasure} \\ \text{prob}}}{p}\, \underbrace{|E\rangle\langle E|}_{\substack{\text{error state ortho} \\ \text{to input space}}}$

Compare $\chi(E_p)$ to the classical analogue.

Note: E_p not unital.

Properties (cont'd):

(5) The most distinguishable inputs need not be optimal.

ie nonortho ρ_x's may be needed.

- Fuchs PRL 79 (1997) 1162 found the first channel that requires nonorthogonal optimal ensemble
- Amplitude damping channel shares the same property (9912122).

Amplitude damping channel:

$$\mathcal{N}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger$$

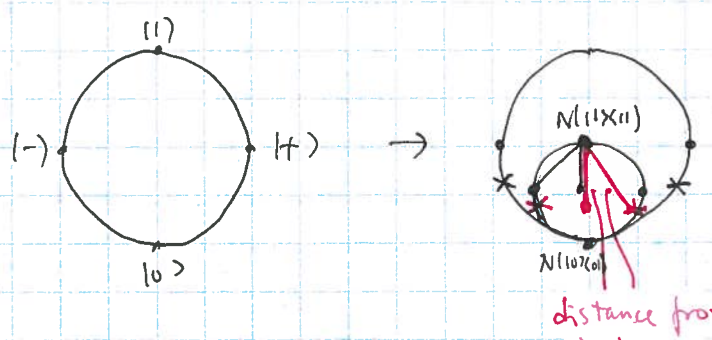
$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$$

$$a|0\rangle + b|1\rangle \xrightarrow{A_0} a|0\rangle + \sqrt{1-\gamma}\, b|1\rangle$$

$$a|0\rangle + b|1\rangle \xrightarrow{A_1} \sqrt{\gamma}\, b|0\rangle$$

If $|0\rangle, |1\rangle$ are energy eigenstates,
ground, excited

then AD represents de-excitation.



distance from original should be small for the average, large for individual.

$\gamma = \frac{1}{2}$

- Restricting to $|\psi_0\rangle$ & $|\psi_1\rangle$, max @ $p_0 = p_1 = \frac{1}{2}$, $\chi = 0.4567$.
- Instead, $\chi = 0.47117$ for some $|\psi_0'\rangle$, $|\psi_1'\rangle$ with $\langle \psi_0' | \psi_1 \rangle \approx \cos 80^\circ$.

How hard is it to calculate or estimate $\chi^{(1)}(N)$?

Beigi & Shor 0707.2090:

Let $C \in \mathbb{R}$. To decide whether $\chi^{(1)}(N) > C$ or $< C - \varepsilon$ for $\varepsilon = \frac{1}{\mathrm{poly}(d)}$ (d = input dim) is NP complete.

NB The above does not imply hardness of estimating $\chi(N^{\otimes n})$.

more structure

④ $\chi^{(r)}(N)$ vs $\chi^{(1)}(N)$

Def: χ strongly additive on N iff $\forall N'$, $\chi^{(1)}(N\otimes N') = \chi^{(1)}(N)+\chi^{(1)}(N')$.

Def: χ weakly additive on N iff $\chi^{(n)}(N) = \chi^{(1)}(N)$

NB: $\forall N, N'$, $\chi^{(1)}(N\otimes N') \geq \chi^{(1)}(N) + \chi^{(1)}(N')$

Pf: Let $\{p_x, \rho_x\}$ and $\{q_y, \sigma_y\}$ be the optimal ensembles for N & N' respectively.

Then $\chi^{(1)}(N\otimes N')$

$$\geq \chi\left(\{p_x q_y, N\otimes N'(\rho_x\otimes\sigma_y)\}\right)$$ ← The product ensembles gives a lower bound for $\chi^{(1)}(N\otimes N')$.

$$= S(XY : BB')$$

$$\Lambda = \sum_{xy} p_x q_y \, |x\rangle\langle x|_X \otimes |y\rangle\langle y|_Y \otimes N(\rho_x)_B \otimes N'(\sigma_y)_{B'}$$

$$= \left(\sum_x p_x |x\rangle\langle x|_x \otimes N(\rho_x)_B\right) \otimes \left(\sum_y q_y |y\rangle\langle y|_y \otimes N'(\sigma_y)_{B'}\right)$$

$$= S(\overset{\text{product}}{\widehat{BB'}}) - S(BB'|XY)$$

$$= S(B)+S(B') - \underbrace{\sum_{xy} p_x q_y \underbrace{S(N(\rho_x)\otimes N'(\sigma_y))}_{S(N(\rho_x))+S(N'(\sigma_y))}}_{\sum_x p_x S(N(\rho_x)) + \sum_y q_y S(N'(\sigma_y))}$$

$$= \chi(\{p_x, N(\rho_x)\}) + \chi(\{q_y, N'(\sigma_y)\})$$

$$= \chi^{(1)}(N) + \chi^{(1)}(N')$$

Thus additive $\Leftrightarrow$ optimal ensemble for $N \otimes N'$ is the product of optimal ensembles for N & N'

$\Leftrightarrow$ optimal ensemble contains product states only

NB Strongly additive $\Rightarrow$ weakly additive.

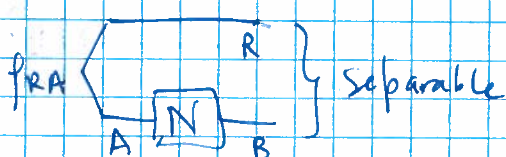
Known additivity result:

(1) If N is entanglement breaking, then χ is strongly additive on N (0302031)

Cor: $C(N) = \chi^{(1)}(N)$.

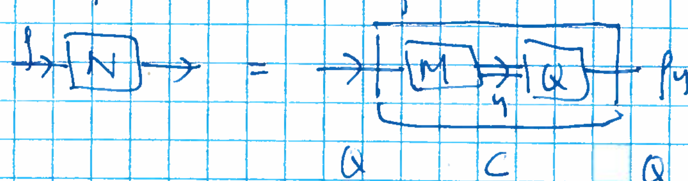
Def: N is entanglement breaking

if $\forall \rho_{RA}$, $I \otimes N(\rho_{RA})$ is separable (convex combination of product states)



Characterization: N entanglement breaking $\Leftrightarrow$ N = QCQ channel

ie



Pf idea: N ent breaking $\Leftrightarrow$ $J(N)$ separable $\Leftrightarrow$ N is QCQ.

Choi state

NB: Classical channels & Q-boxes are entanglement breaking.

Insight: Entangled ensembles & entangled output (from diff uses of N) needed for superadditivity.

② If N unital (ie $N(I)=I$) and N
has 2 dim input & output,
then χ is strongly additive on N.
(King 0103156)
Cor: $C(N) = \chi^{(1)}(N)$ for qubit unital channels

Aside: all qubit unital channels have the form:

$$\rightarrow \boxed{U} - \boxed{\mathcal{E}} - \boxed{V} \rightarrow$$

where $U, V \in \mathbb{U}(2)$, $\underbrace{\mathcal{E}(\rho) = \sum_i p_i \, \sigma_i \rho \sigma_i^\dagger}_{\text{random Pauli channel}}$

③ If N is the d-dim depolarizing channel
then χ is strongly additive on N.
Cor: $C(N) = \chi^{(1)}(N)$.

<u>Unknown additivity:</u>
Most channels, including the amplitude damping channel N_γ
So $C(N_\gamma)$ unknown...

<u>Non additivity:</u>
$\exists N_1, N_2$ s.t. $\chi^{(1)}(N_1 \otimes N_2) \underset{\neq}{>} \chi^{(1)}(N_1) + \chi^{(1)}(N_2)$

Hastings 0809.3972

Def: Entanglement of formation

$$E_f(\rho) = \min_{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|} \sum_i p_i \underbrace{E(|\psi_i\rangle)}_{S(\mathrm{tr}_A |\psi_i\rangle\langle\psi_i|)}$$

ρ given

$p_i, |\psi_i\rangle$ to be optimized under the above constraint

$\sum_i p_i$: average over i

ie $E_f(\rho)$ = min ave entanglement in a convex decomp of ρ into pure states.

Intuition: to make $\rho^{\otimes n}$ for large n

① For each i, Alice & Bob create $\simeq np_i$ copies of $|\psi_i\rangle$ using $np_i E(|\psi_i\rangle)$ ebits & $O(\sqrt{np_i})$ cbits.

② Alice draws i wp p_i, repeats n times iid to obtain $i_1 i_2 \cdots i_n$ and shares outcome with Bob.

③ They place a copy of $|\psi_{i_1}\rangle$ in $A_1 B_1$

$|\psi_{i_2}\rangle$ -- $A_2 B_2$

$\vdots$

$|\psi_{i_n}\rangle$ $A_n B_n$

Tracing out the systems holding $i_1 i_2 \ldots i_n$, they obtain $\rho^{\otimes n}$.

Thus the "entanglement of formation" BDSW96, where min over convex decomp of ρ "minimizes" the ebit cost.

Qn: Is $E_f(\rho)$ the min asymptotic # ebits needed for copy of ρ?

In 0008134, Hayden, Horodecki, and Terhal defined the entanglement cost:

$$E_C(\rho) = \lim_{\substack{n\to\infty \\ \epsilon_n\to 0}} \frac{1}{n}\left(\begin{array}{l}\text{\# ebits needed to product } \rho^{\otimes n} \\ \text{with accuracy } \epsilon_n \text{ \& w/ free cbits}\end{array} \right)$$

and proved:

$$E_C(\rho) = \lim_{n\to\infty} \frac{1}{n} E_f(\rho^{\otimes n}) \leq E_f(\rho)$$

$\neq$ if $\rho^{\otimes n} = \sum_j q_j |\phi_j\rangle\underbrace{\langle\phi_j|}_{\text{entangled over } A_1B_1A_2B_2 \dots A_nB_n}$

gives decomposition better than

$\rho^{\otimes n} = \left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right)^{\otimes n}$

one copy in each of $(A_1B_1)(A_2B_2)\dots(A_nB_n)$

all true or all false

In 0305035, Shor prove TFAE:

① $\forall N_1, N_2 \quad \chi^{(1)}(N_1 \otimes N_2) = \chi^{(1)}(N_1) + \chi^{(2)}(N_2)$

② $\forall \rho_1, \rho_2, \quad E_F(\rho_1 \otimes \rho_2) = E_F(\rho_1) + E_F(\rho_2)$

③ $\forall N_1, N_2, \quad S_{min}(N_1 \otimes N_2) = S_{min}(N_1) + S_{min}(N_2)$ ← what Hastings disproved. (Lec 18 S 2010)

[where $S_{min}(N) = \min_\rho S(N(\rho)) = $ min output entropy]

④ $\forall \rho_{A_1B_1A_2B_2}, \quad E_F(\rho_{A_1A_2|B_1B_2}) \geq E_F(\rho_{A_1B_1}) + E_F(\rho_{A_2B_2})$

(Lecs 8-9 S 2010)

How was ③ disproved?

References: Hastings 0809.3972

- Also Brandao, (M Horodecki 0907.3210
  Fukuda, King, Mosor 0905 3697
  Aubrun, Szarek, (E) Werner 0910.1189.

- Built on failur of ③ for Renyi-entropies
  Holevo-Werner (R, Winter, Hayde

Main ideas:

- Let input dim = output dim = d.
- Choose N_1: $N_1(\rho) = \frac{1}{n}\sum_{k=1}^{n} U_k \rho U_k^\dagger$, $U_k \in \mathcal{U}(d)$

  $N_2 = \overline{N}_1$: $N_2(\rho) = \frac{1}{n}\sum_{k=1}^{n} \overline{U}_k \rho \overline{U}_k^\dagger$. "‾" complex conjugate

- Show $S_{min}(N_1 \otimes N_2) \lesssim 2\log n - \frac{\log n}{n}$

  Reason:

  Joint input $|\Phi\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} |i\rangle|i\rangle$ well preserved.

  Reason:

  Transpo trick: $A \otimes I|\Phi\rangle = I \otimes A^T|\Phi\rangle$ $\forall$ d×d ces

  Pf - ~~Ex (1 (n~~).

  Thus $U \otimes \overline{U}|\Phi\rangle = I \otimes U^T \overline{U}|\Phi\rangle = |\Phi\rangle$.

$\therefore$ n out of n^2 Kraus ops in $N_1 \otimes N_2$ fix $|\Phi\rangle$

$\therefore$ $\langle \Phi | \, N_1 \otimes N_2 (|\Phi\rangle\langle\Phi|) \, |\Phi\rangle \geq \frac{1}{n}$.

Worse case spectrum : $\{ \frac{1}{n}, \underbrace{(1-\frac{1}{n})\frac{1}{n^2-1}, \; \ldots \; (1-\frac{1}{n})\frac{1}{n^2-1}}_{n^2-1 \text{ times}} \}$.

(Note $N_1 \otimes N_2 (|\Phi\rangle\langle\Phi|)$ rank $\leq n^2$.)

$$\Rightarrow S_{min}(N_1 \otimes N_2) \leq S(N_1 \otimes N_2 (|\Phi\rangle\langle\Phi|))$$

$$\leq -\frac{1}{n}\log\frac{1}{n} - (1-\frac{1}{n}) \log (1-\frac{1}{n^2})\left(\frac{1}{n^2-1}\right)$$

$$\leq 2\log n - \frac{\log n}{n}.$$

- Note $S_{min}(N_1) = S_{min}(N_2)$.

  Since $S(N_1(\rho)) = S(N_2(\bar{\rho}))$

  Difficult part:

  $$\forall |\psi\rangle, \; S(N_1(|\psi\rangle\langle\psi|)) \geq \log n - \left[\frac{\text{const}}{n} + \text{poly}(n) \, O\sqrt{\frac{\log d}{d}} \right]$$

  Requires choosing U_k's according to Haar measure.

  Assume $\exists$ input with low entropy output & get contradiction

- Choose $d \ggg n$ gives $S_{min}(N_1 \otimes N_2) \lneq 2 \, S_{min}(N_1)$.

Note large dim, nonconstructive, variations on N_1, & proof of method.
tight race of 2 terms.