QIC890/CO781 lec 17 Nov 10, 2016

Last time:

- $I_c(R\rangle B)_\rho := [S(B) - S(RB)]_\rho = [S(B) - S(E)]_{|\alpha\rangle}$ where $\mathrm{tr}_E |\alpha\rangle\langle\alpha|_{RBE} = \rho_{RB}$

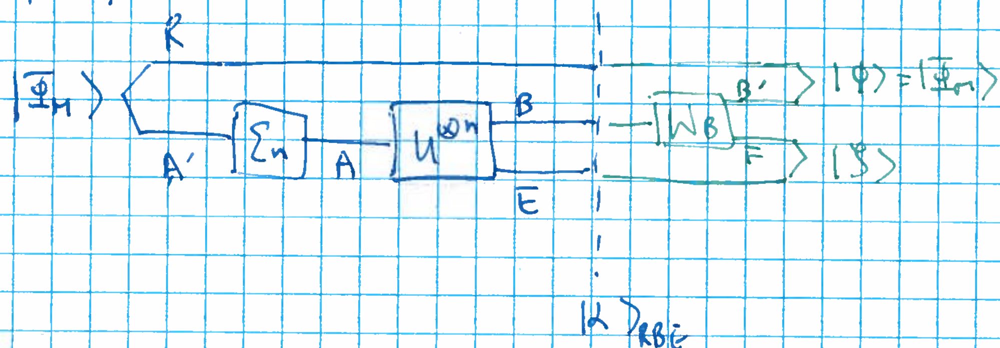
- $Q^{(1)}(N) = \max_{|\psi\rangle} I_c(R\rangle B)_{I\otimes N(|\psi\rangle\langle\psi|)} = \max_{|\psi\rangle} [S_B - S_E]_{|\alpha\rangle = I\otimes U|\psi\rangle}$

  U: iso ext of N

- $Q^{l}(N) = \frac{1}{l} Q^{(l)}(N)$

- LSD Thm: $Q(N) = \sup_l Q^{(l)}(N)$

- Decoupling:



$\rho_{RE} = \left(\frac{I}{M}\right)_R \otimes \rho_E \iff \exists\, W_B$ s.t. $(W_B \otimes I_{RE})|\alpha\rangle_{RBE} = |\Phi_M\rangle_{RB'} \otimes |\xi\rangle_{EF}$

R decoupled from E

$\mathcal{E}_n$ is a code transmitting $|\Phi_M\rangle$ perfectly.

$\Rightarrow$ Approx version

Approx decoupling $\Rightarrow$ approx transmission of $|\Phi_M\rangle$

Lemma:

If $\left\| \rho_{RE} - \left(\frac{I}{M}\right)_R \otimes \rho_E \right\|_1 \leq \epsilon'$

then $\exists\, |\alpha\rangle_{REB}$ purifying ρ_{RE}, for $B = B'F$

s.t. $\left\| \mathrm{tr}_{EF}\, |\alpha\rangle\langle\alpha|_{REB} - |\Phi_M\rangle\langle\Phi_M|_{RB'} \right\|_1 \leq 2\sqrt{\epsilon'}$

Pf: (tr dist $\Rightarrow$ F $\Rightarrow$ F of purification $\Rightarrow$ tr dist of purifications $\Rightarrow$ tr dist)

- Recall $1 - F(\mu,\nu) \leq \frac{1}{2}\|\mu - \nu\|_1$ ($\forall\, \mu,\nu$ states)

$\therefore\ F\left(\rho_{RE}, \left(\frac{I}{M}\right)_R \otimes \rho_E\right) \geq 1 - \frac{\epsilon'}{2}$

since any $|\beta\rangle = U \otimes I\, |\beta_0\rangle$, absorb U into $|\alpha\rangle$

- Recall [Uhlmann] $F(\mu,\nu) = \max_{\substack{|\alpha\rangle,|\beta\rangle \\ \text{purifies } \mu \quad \text{purifies } \nu}} |\langle\alpha|\beta\rangle| = \max_{\substack{|\alpha\rangle \\ \text{purifies } \mu}} |\langle\alpha|\beta_0\rangle|$ ($|\beta_0\rangle$: fixed purification of ν.)

- Choose $|\beta_0\rangle = |\Phi_M\rangle_{RB'} \otimes |\xi\rangle_{EF}$ where $\nu = \left(\frac{I}{M}\right)_R \otimes \rho_E$, $\mu = \rho_{RE}$

Then: $\exists\, |\alpha\rangle_{REB}$ attaining the max in $F(\mu,\nu)$

$\therefore\ 1 - \frac{\epsilon'}{2} \leq F\left(\rho_{RE}, \left(\frac{I}{M}\right)_R \otimes \rho_E\right) = F\left(|\Phi_M\rangle_{RB'} \otimes |\xi\rangle_{EF}, |\alpha\rangle_{REB}\right)$

- Recall $\frac{1}{2}\|\mu - \nu\|_1 = \sqrt{1 - F(\mu,\nu)^2}$ if μ,ν pure. Apply to the 2 purifications:

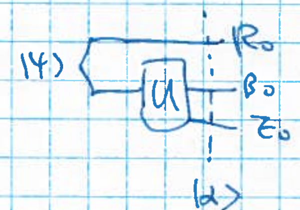
$\therefore\ \left\| |\alpha\rangle\langle\alpha|_{REB} - |\Phi_M\rangle\langle\Phi_M|_{RB'} \otimes |\xi\rangle\langle\xi|_{EF} \right\|_1$

$= 2\sqrt{1 - F\left(|\alpha\rangle_{REB}, |\Phi_M\rangle_{RB'} |\xi\rangle_{FE}\right)^2}$ $\quad [(1-f^2) = (1-f)(1+f)]$

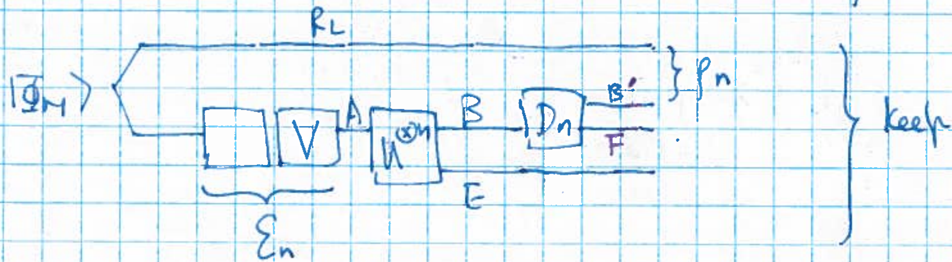
$\leq 2\sqrt{2}\sqrt{1-F} \leq 2\sqrt{\epsilon'}$. Finally, Tracing out EF, $\| \ \|_1$ non increasing.

Direct coding thm:

- Let $\mathcal{U}$ = isometric extension of $\mathcal{N}$.
- For any $|\psi\rangle_{R_0 A_0}$, let $|\omega\rangle = \underbrace{I_{R_0} \otimes \mathcal{U}(|\psi\rangle)}_{\text{lives on } R_0 B_0 E_0}$



- We show $\exists (M,n)$ code achieving the following:

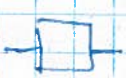


$A = A_1 \cdots A_n$, $B = B_1 \cdots B_n$, $E = E_1 \cdots E_n$

$M \geq 2^{n(r-\delta_n)}$, $\| \rho_n - \Phi_M \|_1 \leq \epsilon_n$

$r = I_c(R_0 \rangle B_0)_{|\omega\rangle}$

- We consider a random choice of codes where

 embeds $\mathbb{C}^M$ into $A = A_1 \cdots A_n$

 is drawn according to the Haar measure, $V \in \mathcal{U}(A_1 \cdots A_n)$

keep on board ? ⊙ To upper bound the error ϵ_n, we show the state on R_L & E … (I.) … … mplies

$\exists D_n$ s.t … y B'

and E …

Requires … ations

Then from decoupling lemma

Called the decoupling approach to showing the direct coding thm

② why on average over V, R_L & E decoupled approx

Together $\exists V$ s.t. R_L & E decoupled enough so R_L purified by B' resulting in the desirable Φ_M on R_L & B'.

(by lemma)

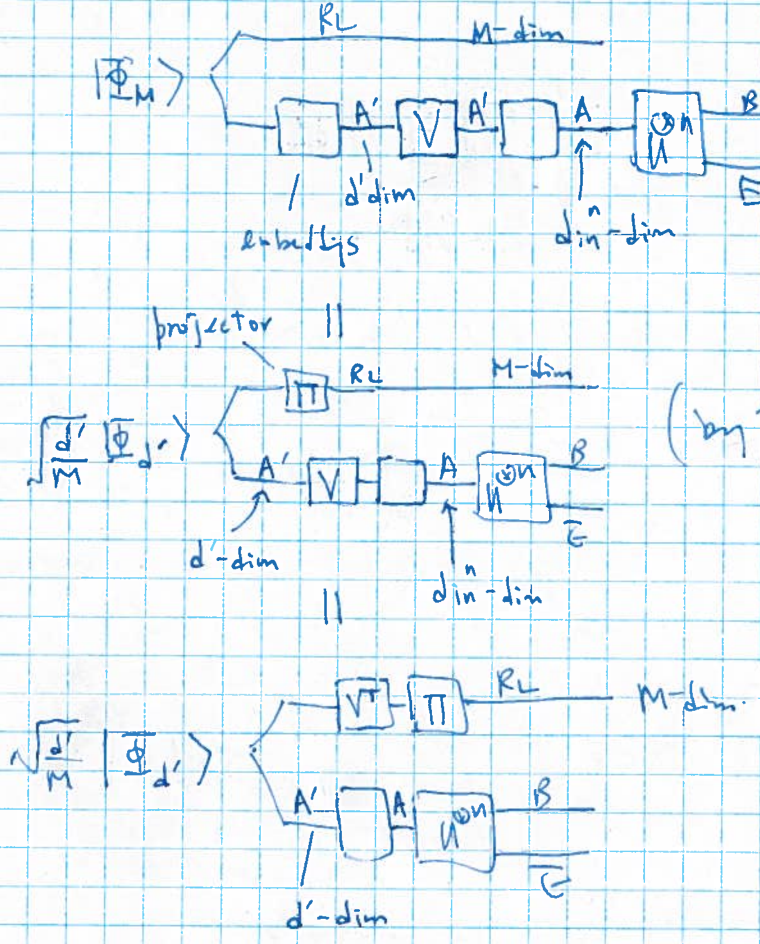
- Replace N by $N^{\otimes \ell}$ and repeat the above argument gives the existence of codes attaining rate $Q^{(\ell)}(N)$ under the error meas (1b) (for creating max entangled states)

- Finally, apply 9809010 / 0311037 (trimming the code space) $Q^{(\ell)}(N)$ is achievable for error meas (1a).

② Why on average over V, R_L & E decoupled approx:

summary

I



(by transpose trick)

Lemma: (0702005 Thm III) (Use Green)

Let ① ρ^{SE} be max mixed on S

② $P^{S\to R}$ be a projector into subspace $R \subseteq S$

③ $\psi_u^{RE} := \frac{|S|}{|R|} (PU)^S \otimes I^E \, \rho^{SE} \, (U^\dagger P)^S \otimes I^E$

Then: ① $\| \psi_U^{RE} - \frac{I_R}{|R|} \otimes \psi^E \|_2^2 = \operatorname{Tr}(\psi_u^{RE})^2 - \frac{1}{|R|} \operatorname{Tr}(\psi^E)^2$

② $\int dU \, \| \psi_u^{RE} - \frac{I_R}{|R|} \otimes \psi^E \|_2^2 \leq \operatorname{Tr}(\rho^{SE})^2$

③ $\int dU \, \| \psi_u^{RE} - \frac{I_R}{|R|} \otimes \psi^E \|_1^2 \leq |RE| \operatorname{tr}(\rho^{SE})^2$

$|RE|$: $|R| = \dim R$, $|E| = \dim E$

Pf: see supp notes on course website.

(or Watrous book)

① $\| A - B \|_2^2 = \operatorname{Tr} (A-B)^\dagger (A-B)$

$= \operatorname{tr}(A^\dagger A - B^\dagger A - A^\dagger B + B^\dagger B)$

② $\int dU \operatorname{Tr}(\psi_u^{RE})^2 = \underline{\underline{\int dU}} \operatorname{tr} \left(\underbrace{\psi_u^{RE} \otimes \psi_u^{RE}}\right) \underset{\text{SWAP}}{F^{RE \leftrightarrow \hat{R}\hat{E}}}$

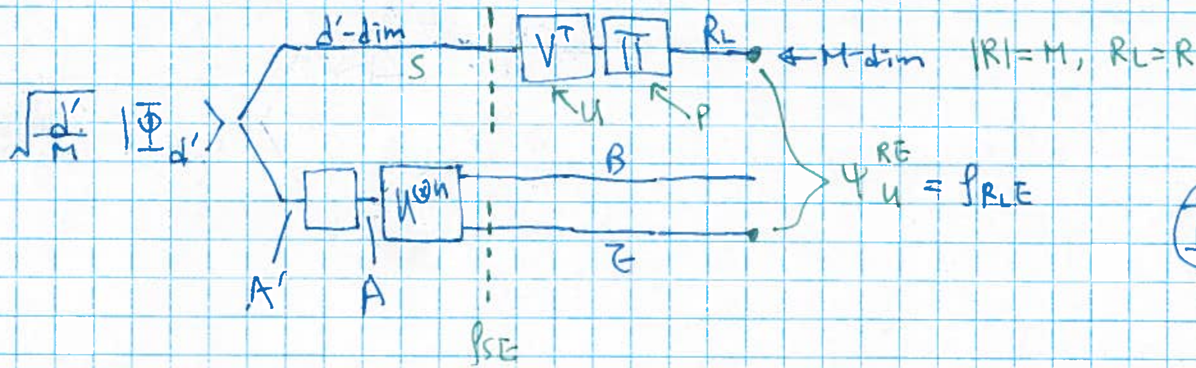
$\int dU \operatorname{tr} (P\otimes P) \underline{U \otimes U} [\rho^{SE} \otimes \rho^{\tilde{S}\tilde{E}}] \underline{U^\dagger \otimes U^\dagger} (P \otimes P) F^{RE \leftrightarrow \hat{R}\tilde{E}}$

$= \operatorname{tr}(\rho^{SE} \otimes \rho^{\tilde{S}\tilde{E}}) \cdot \boxed{\;///\;}$

③ $\| M \|_1^2 \leq (\dim M) \| M \|_2^2$ Cauchy-Schwarz ineq.

$(\lambda_1 + \cdots + \lambda_d)^2 \leq d (\lambda_1^2 + \cdots + \lambda_d^2)$

Identifying objects in lemma w/ decoupling problem:



(if LHS is $|\Phi_{d'}\rangle$ w/o the factor $\sqrt{\frac{d'}{M}}$)

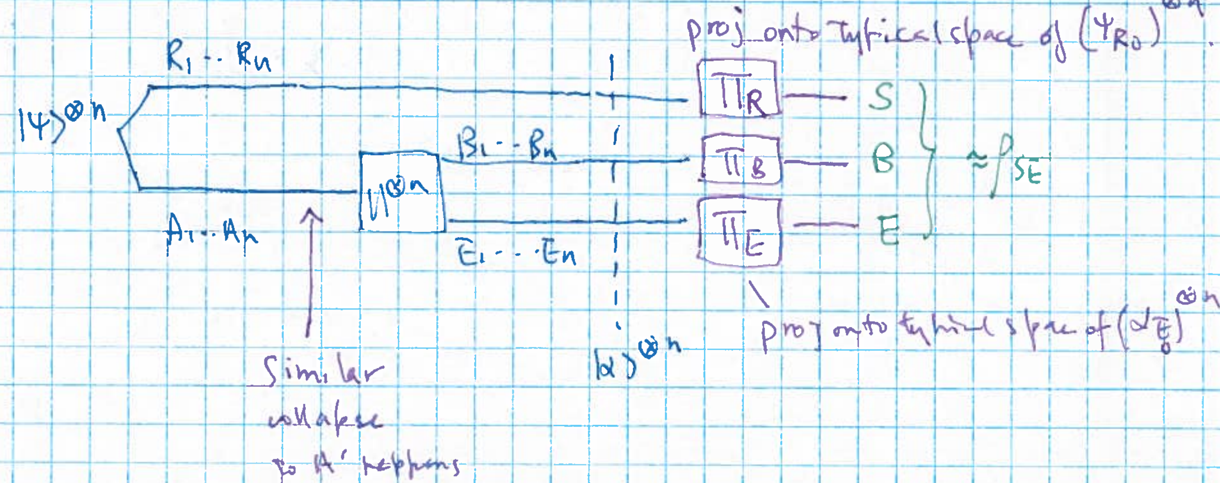
So conclusion ① gives:

$$\int dV^T \ \left\| \rho_{R_L E} - \left(\frac{I}{M}\right)_{R_L} \otimes \rho_E \right\|_1^2 \overset{(\ddagger)}{\leq} M \cdot |E| \ tr\left(\rho^{SE}\right)^2$$

* Intuition what M is achievable:

Replace the state @ the green dotted line (whose reduction on SE is ρ_{SE}) by the following (as a first step):

$d' \approx 2^{nS(R_0)_\alpha}$



$\left(tr\left(\frac{I}{d}\right)^2 = \frac{1}{d} \right)$

Then ρ_{SE} "almost" max mixed on S, Π_R, Π_B, Π_E succeeds whp, and state largely unchanged (see P1, lec 07, part 2)

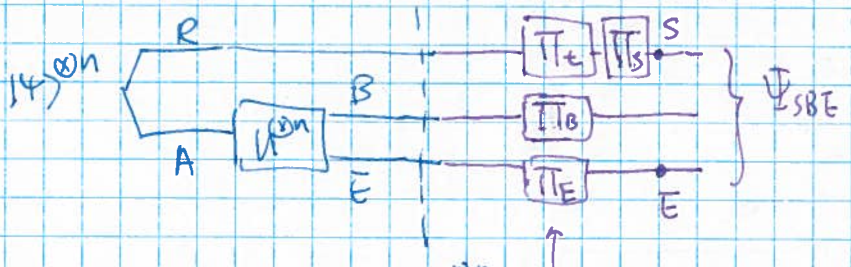
Then, $\dim E \approx 2^{nS(E_0)_\alpha}$, $tr(\rho_{SE})^2 = tr(\rho_B)^2 \approx 2^{-nS(B_0)_\alpha}$ $\approx$ max mixed.

$\therefore$ $M \leq 2^{n[I_c(R_0 \rangle B_0)_\alpha - \delta_n]}$ makes RHS of $(\ddagger)$ vanish.

[This also defines the whole space]

Need a better state Ψ_{SBE} to apply Thm 3

proj onto a type class :



proj onto strongly typical spaces.

Let $\Psi_R = \sum_i \lambda_i |i\rangle\langle i|$

type class t for $\Psi_R^{\otimes n}$ contains all $|i_1 \cdots i_n\rangle$ with same empirical freq of the symbols.

Π_t = proj onto span of elts in the type class.

Let A_t be the space spanned.

or Lemma 8.55 Watrous

Thm 4: 0702005 $\quad \forall\, \epsilon, \delta > 0$

$\exists\, n$ large enough s.t. $\exists$ type class t for $\Psi_R^{\otimes n}$ s.t.

$$\exists\, S \subset A_t , \quad \dim(S) \ge \left(1 - \frac{\sqrt{\epsilon}}{2}\right) \dim(A_t)$$

- $\left\| \frac{(\Pi_S \otimes I_B \otimes I_E)\, |\alpha\rangle^{\otimes n}}{\sqrt{\text{normalization}}} - \Psi_{SBE} \right\|_1 \le \epsilon$

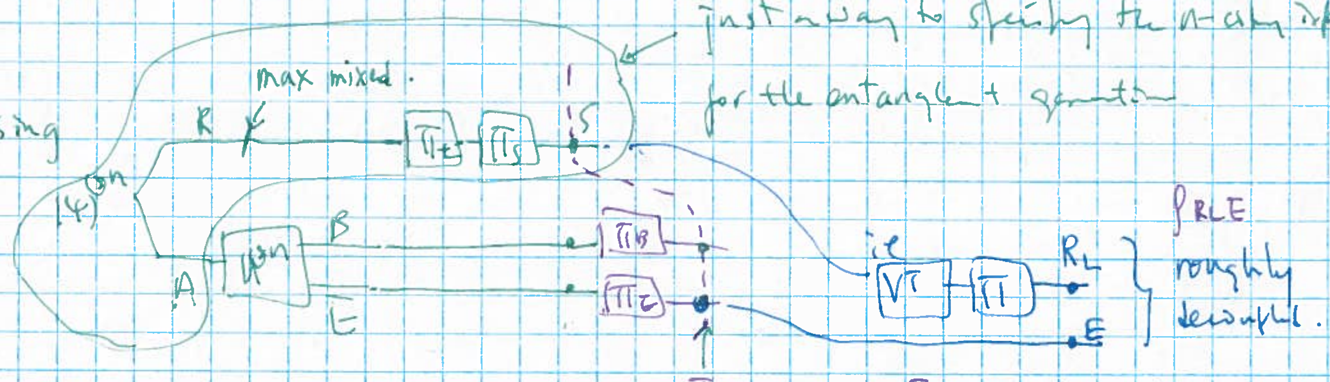
  w/ max mixed S

- $\mathrm{Tr}\left[(\Psi_B)^2 \right] \le 2^{-n(S(B_0)_\alpha - \delta)}$
- $(\dim E)_\Psi \le 2^{n(S(E_0)_\alpha + \delta)}$
- $\dim(A_t) \ge 2^{n(S(R_0)_\alpha - \delta)}$

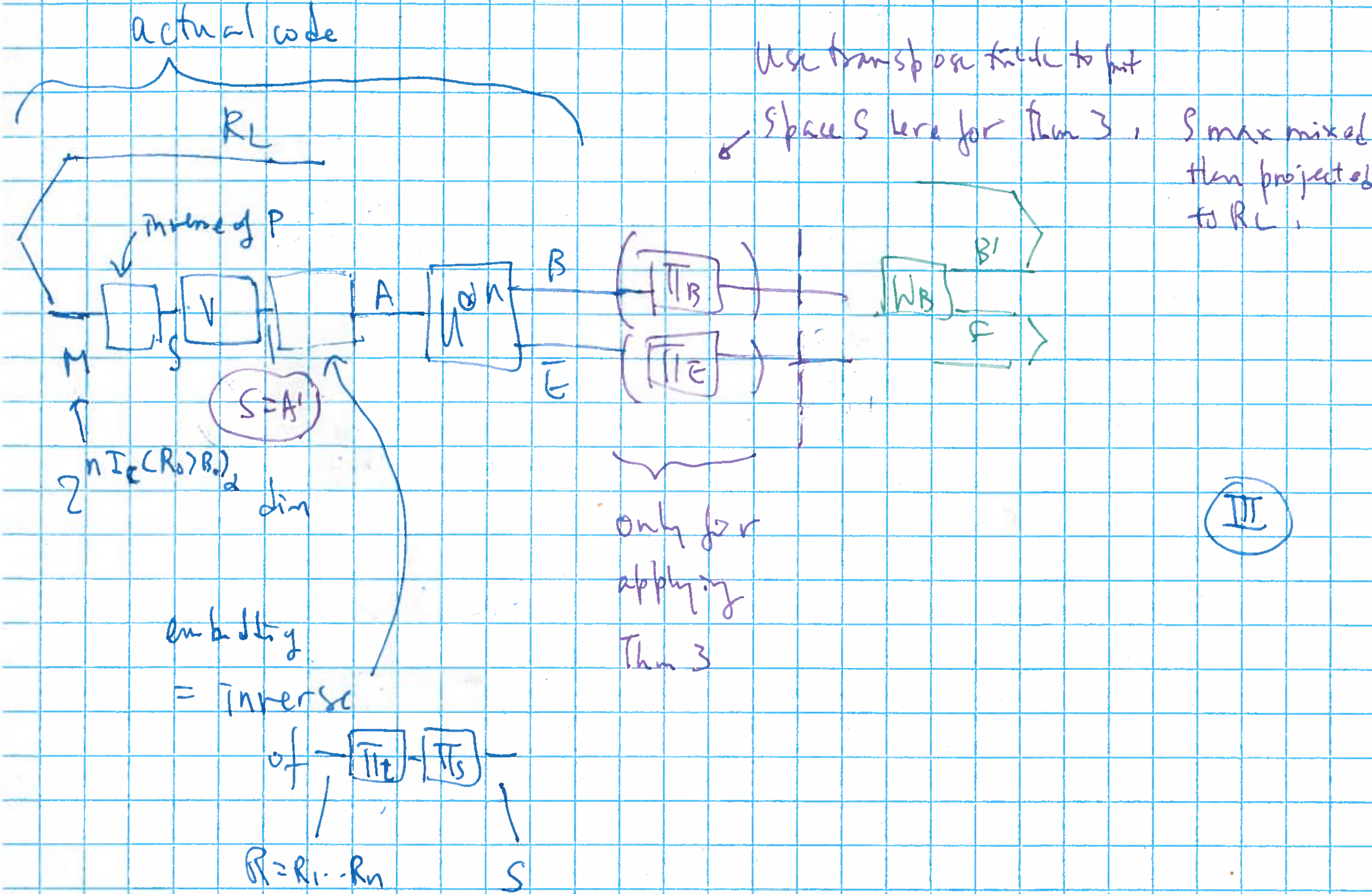
$\boxed{\Psi_{SBE} : \text{carries the desirable spectral / entropic properties of } |\alpha\rangle^{\otimes n}.}$

So using

just a way to specify the n-copy input for the entanglement generation



Ψ_{SBE}, $\mathrm{tr}_B \Psi_{SBE} = \rho_{SE}$ upon which Thm 3 is applied.



These choices allow the correct rate based on [12] simultaneously, with threading, allows Thm 3 to apply.

III

$$\therefore \int dV^{T} \left\| \rho_{R_L E} - \left(\frac{I}{|M|}\right)_{R_L} \otimes \rho_E \right\|_1^2 \leq 2^{n\left(I_c(R_0 \rangle B_0)_\alpha - 3\delta_n\right)} \times 2^{n\left(S(E_0)_\alpha + \delta_n\right)} \times 2^{-n\left(S(B_0)_\alpha - \delta_n\right)}$$

$$\leq 2^{-n \cdot \delta_n} \to 0 .$$

$\therefore \exists V$ s.t. $\left\| \rho_{R_L E} - \left(\frac{I}{|M|}\right)_{R_L} \otimes \rho_E \right\|_1^2$ is small, say ϵ_n

$$\Downarrow$$

By lemma, can decode $|\Phi_m\rangle$ with trace distance $2\sqrt{\epsilon_n}$.