CO781 / QIC890 lec 20. Nov 22, 2016

We saw :

- the LSD thm
- degradable channels
- non additivity of $Q^{(1)}(N)$

In fact $Q^{(1)}(N_f) = 0$ but $Q^{(5)}(N_f) > 0$ for the qubit depolarizing channel for some parameter range of f.

Qn: What are our options to understand the Q capacity?

* Check lec19 notes for corrected proof that depolarizing channel has diagonal optimal input.

① Upper bounds on $Q(N)$ through additive extensions. [Smith-Smolin 0712.2471]

<u>Def</u>: For a given channel N, T is called an <u>additive</u> (degradable) extension of N if ① $\exists\, S$ s.t. $N = S \circ T$
② $Q(T) = Q^{(1)}(T)$ (T degradable)

$$-\boxed{N}- \;=\; -\boxed{T}-\boxed{S}-$$

<u>Thm 1</u>: $Q(N) \leq Q^{(1)}(T)$ if T is an additive extension of N.

<u>Pf (#1)</u>: If $\exists\, \mathcal{E}, D$ s.t. $|\psi\rangle -\boxed{\mathcal{E}}-\boxed{N^{\otimes n}}-\boxed{D}- \approx |\psi\rangle$

then $|\psi\rangle -\boxed{\mathcal{E}}-\boxed{T^{\otimes n}}-\underbrace{\boxed{S^{\otimes n}}-\boxed{D}}_{-\boxed{D'}-}- \approx |\psi\rangle$

same approx

So any rate achievable by N is achievable by T (since the (M, n) code for N gives one for T).

<u>Pf (#2)</u>:
$$
\begin{aligned}
Q^{(1)}(T) &= Q(T) \\
&= \sup_n \frac{1}{n} \max_{|\psi\rangle} I(R\rangle B^{\otimes n})_{I\otimes T^{\otimes n}(|\psi\rangle\langle\psi|)} \\
&\underset{\text{QDPI}}{\geq} \sup_n \frac{1}{n} \max_{|\psi\rangle} I(R\rangle B^{\otimes n})_{I\otimes N^{\otimes n}(|\psi\rangle\langle\psi|)} \qquad (N = S\circ T). \\
&= Q(N)
\end{aligned}
$$

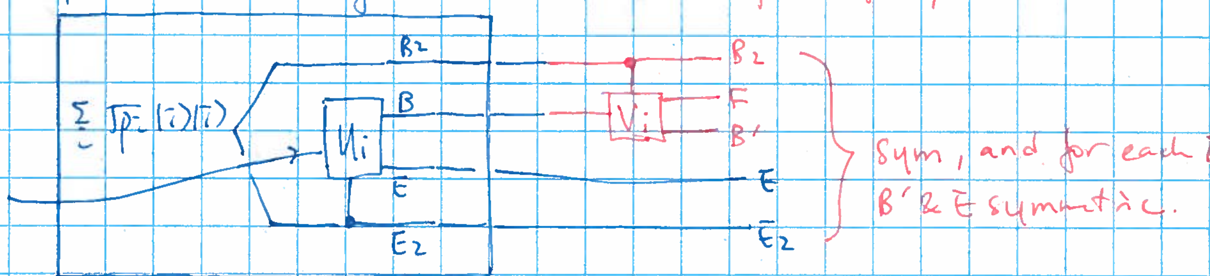
Thm 2 If $N = \sum_i p_i N_i$ and each N_i is degradable

then $Q(N) \le \sum_i p_i Q^{(1)}(N_i)$

Pf: First we show that $T = \sum_i p_i N_i \otimes |i\rangle\langle i|$ is a degradable extension of N.

Let U_i be the iso ext of N_i, V_i be the iso extension of the degradable map of N_i.

A possible iso ext of T is: Possible degrading map



$\therefore$ T is degradable.

Let $S(\rho) = \mathrm{tr}_{B_2}(\rho)$

Then $N = S \circ T$, so T is a degradable extension of N.

Second, we upper bound $Q^{(1)}(T)$:

For any $|\psi\rangle_{RA}$, $I \otimes T(\underbrace{|\psi\rangle\langle\psi|}_{RA}) = \sum_i p_i \underbrace{(I \otimes N_i)(|\psi\rangle\langle\psi|)}_{RB} \otimes |i\rangle\langle i|_{B_2}$

$$I_c(R\rangle BB_2) = \sum_i p_i \, I_c(R\rangle B)_{I\otimes N_i(|\psi\rangle\langle\psi|)}$$

$$\le \sum_i p_i \, Q^{(1)}(N_i)$$

$$\therefore Q^{(1)}(T) = \sup_{|\psi\rangle} I_c(R\rangle BB_2)_{I\otimes T(|\psi\rangle\langle\psi|)} \le \sum_i p_i Q^{(1)}(N_i)$$

Together: $Q(N) \le Q^{(1)}(T) \le \sum_i p_i Q^{(1)}(N_i)$.

Thm 1

eg: $N_q(\rho) = (1-q)\rho + \frac{q}{3}(X\rho X + Y\rho Y + Z\rho Z)$

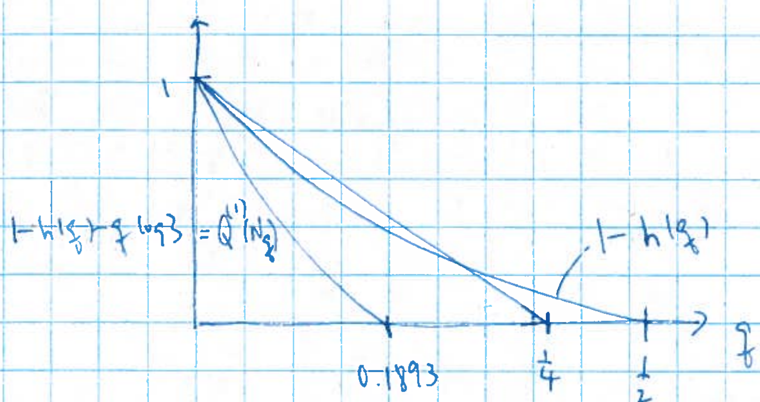
$$= \frac{1}{3}\left(\left[(1-q)\rho + qX\rho X\right] + \left[(1-q)\rho + qY\rho Y\right] + \underbrace{\left[(1-q)\rho + qZ\rho Z\right]}_{N_{Z,q}} \right)$$

Recall $Q^{(1)}(N_{Z,q}) = 1 - h(q)$

$\therefore N_q(\rho) \le \frac{1}{3}(1-h(q)) \times 3 = 1-h(q)$

This is called the Rain's bound, initially proved in

0008047 Cor 5.7.



Thm 3 Let $N = \sum_i p_i N_i$

T_i degradable extension for each N_i.

Then $Q(N) \leq \sum_i p_i Q^{(1)}(T_i)$.

NB Thm 3 generalizes Thm 2 to arbitrary convex decomposition of N.

It also shows that, the convex combination of upp bdds obtained from degradable extensions of a convex decomp of N is an upper bound of $Q(N)$.

Pf: First note that $T = \sum_i p_i T_i \otimes |i\rangle\langle i|$ is a degradable extension of N.

Second, $Q^{(1)}(T) \leq \sum_i p_i Q^{(1)}(T_i)$.

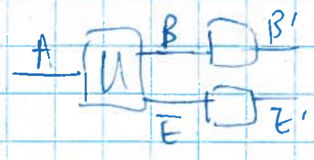
Both proofs similar to Thm 2. Will do first proof in A4.

<u>Thm 4</u> If N antidegradable,

then N has a zero capacity degradable extension.

Pf: let U be the iso ext of N

$d = \max(d_E, d_B)$, and embed B into B', E - - - E' } both d-dim.



Define isometric $V: A \to B'E'C_1C_2$ as:

$$V|\phi\rangle = \frac{1}{\sqrt{2}}\left(U|\phi\rangle\right)_{B'E'} |01\rangle_{C_1C_2} + \frac{1}{\sqrt{2}}\left(\mathrm{SWAP}_{B'E'}\, U|\phi\rangle\right) |10\rangle_{C_1C_2}$$

Let $T(\rho) = \operatorname{tr}_{E'C_2} V\rho V^\dagger$.

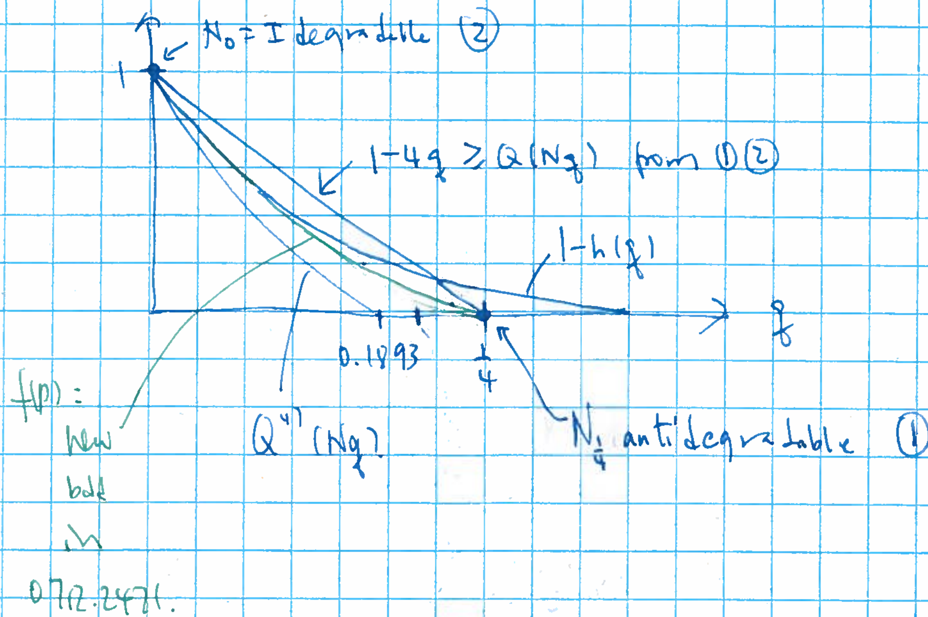
Then T degradable (& antidegradable), $Q(T) = 0$

op S { Also, Bob can condition on C_1 being 1, degrade sys B' in $\left(\mathrm{SWAP}_{B'E'}\, U|\phi\rangle\right)$ to sys E', which is sys B' in $U|\phi\rangle$.

Together $N = S \circ T$.

$\therefore$ T is a degradable extension of N.

$Q(N) \le Q^{(1)}(T) = 0$.



Also can use AD channel as a degradable extension

$$A_\gamma(\rho) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \rho \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} \rho \begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix}$$

Where $\gamma = 4\sqrt{1-p}\,(1-\sqrt{1-p})$.

Then $Q(N_p) \leq H\left(\frac{1-\gamma}{2}\right) - H\left(\frac{\gamma}{2}\right)$.

Together: $f(p) =$ max convex fcn upper bounded by

$1-H(p)$, $1-4p$, $H\left(\frac{1-\gamma}{2}\right) - H\left(\frac{\gamma}{2}\right)$.

Then Thm 4 $\Rightarrow$ $Q(N) \leq f(p)$

② Continuity of $Q(N)$ [0810.4931, L. Smith] ← relies on Fannes-Alicki

If N, M are channels from A to B

and $\| N-M \|_\diamond \leq \epsilon$

then, $|Q(N)-Q(M)| \leq 8\epsilon \log(\dim B) + 4h(\epsilon)$

(Similar results hold for $C(N)$ & $P(N)$.)

classical, private capacities. Improved F-A ineq. in 1507.07775

③ Approx degradable channels. [1412.6980 Sutter, Scholz, Winter, Renner]

* Ala's presentation

④ Assisted capacities.