Theorem: superdense coding (Bennett-Wiesner 93)

Suppose Alice and Bob share the state $\frac{1}{\sqrt{s}}\sum_{i=1}^{s}|i\rangle\otimes|i\rangle$ and Alice can send an s-dimensional quantum system to Bob. Then, Alice can communicate $t=s^2$ messages to Bob!

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How to think about quantum protocols:

Which party has what classical information ?

Which party has what quantum system ?

What operations he/she is allowed to do ?

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How to think about quantum protocols:

Which party has what classical information ?

Alice has a message $v\in\{0,x,y,z\}$. Bob has nothing.

Which party has what quantum system ?

Initially, Alice (Bob) has the 1st register A (B) of the shared state. Alice also has another s-dim system C. She sends C to Bob. Then, Bob has both B and C.

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How to think about quantum protocols:

What operations he/she is allowed to do ?

Before Alice sends C to Bob, she can apply any operation on AC that depends on v. C depends on A and v, and C can be A itself.

After Bob receives C from Alice, he can apply any operation on AC that does not depend on v.

Proof: for simplicity, first consider s=2.

Suppose Alice & Bob share the state $|\Phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

so that Alice (Bob) holds the first (second) qubit A (B).

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Recall the Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Suppose Alice wants to communicate a message v from the set $\{0, x, y, z\}$.

If her message is v, she applies σ_v to A.

The shared state $|\Phi_0\rangle$ on AB is transformed by $\sigma_v \otimes I$.

For $|\Phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\Phi_0\rangle = \sigma_0 \otimes I \, |\Phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\Phi_x\rangle = \sigma_x \otimes I \, |\Phi_0\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$$

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These 4 states are mutually orthogonal, forming the "Bell basis". Note that Alice operates on a 2-dim system A, but the shared state on AB traverses to 1 out of 4 possible distinguishable (ortho) states.

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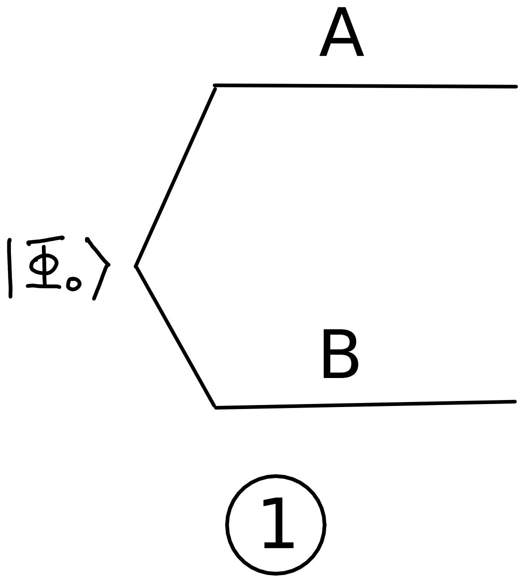
If Alice sends C=A to Bob, he has AB in the state $|\Phi_v\rangle$.

He can measure AB along the Bell basis to find v !

Communication protocol:



Initial state shared between Alice and Bob. Alice is holding system A; Bob is holding system B.

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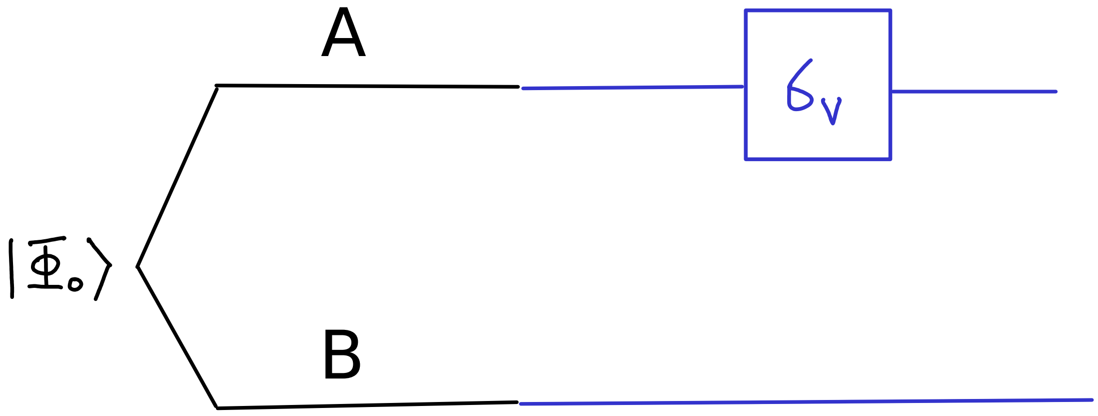


①

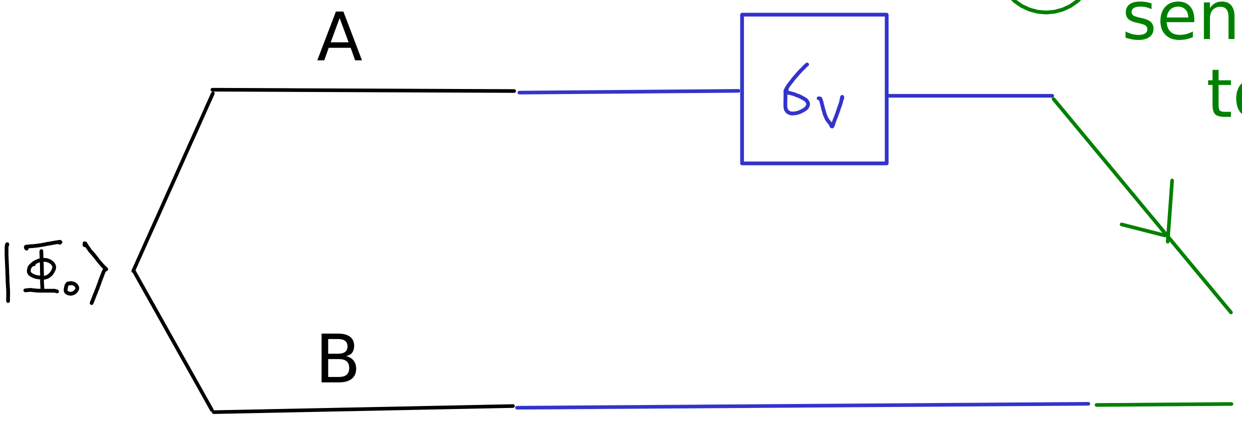
Initial state shared between Alice and Bob. Alice is holding system A; Bob is holding system B.

②

If Alice wants to communicate "v" $\in$ {0,x,y,z} to Bob she applies σ_v to qubit A.

(4 possibilities)

Communication protocol:



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(4 possibilities)

③ Alice sends system A to Bob (2-dim).

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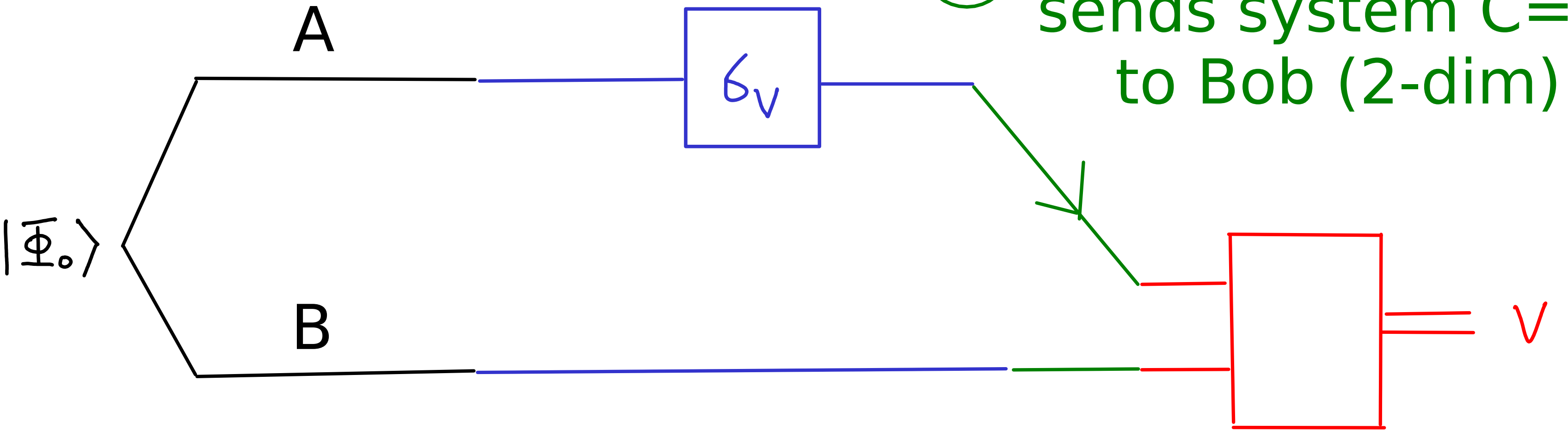

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(4 possibilities)

③ Alice sends system C=A to Bob (2-dim).

④ Having both systems A & B, Bob measures along the Bell basis. Outcome is v with certainty.

Thoughts:

1. Entanglement enables the operation on a 2-dim system to map the shared state over 4 dimensions.

2. Bob has a 4-dim system (AB) after the channel transmission, so superdense coding is consistent with Holevo's bound.

3. Is there a catch? Does Alice also need to prepare the entangled state in AB and send B to Bob before superdense coding so altogether she sends 4 dims?

   Not really. Bob can prepare the entangled state in AB and send A to Alice instead, or a common friend Charlie can prepare the entangled state and send A to Alice and B to Bob.

   SD turns entanglement or back quantum comm into increased forward classical communication !!

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Converting the units of various resources:

s-dim quantum state = log s qubits
s^2 classical messages = 2 log s bits
max entangled state of local dim s = log s "ebits"

$$\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right)_{A_1B_1}\otimes\cdots\otimes\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right)_{A_nB_n}=\frac{1}{\sqrt{2^n}}\sum_{u\in\{0,1\}^n}\underbrace{|u\rangle}_{A_1\cdots A_n}\otimes\underbrace{|u\rangle}_{B_1\cdots B_n}$$

Dividing everything by log s, on average, SD coding uses 1 ebit and sends 1 qubit to communicate 2 bits (doubling the rate).

What if Alice wants to communicate a quantum state to Bob by sending only classical data?

For simplicity, she wants to communicate a qubit $|\psi\rangle = a|0\rangle + b|1\rangle$ to Bob.

Case (i): Alice knows a,b (she authors the message)

She can send approximations of a and b to Bob. For Bob to decode a qubit closer and closer to $|\psi\rangle$ she has to send more and more bits.

Case (ii): Alice is given the state to be communicated (she runs Qedex, usual setting)

She does not know a,b, and cannot know more than 1 bit of information about them by Holevo's bound.

Can't comm quantum states by sending classical data.

Free entanglement is like free love
-- it changes the world.

Charles Bennett, Cambridge, 1999

Teleportation

Alice can communicate a qubit to Bob
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(2) they share the ebit $|\Phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

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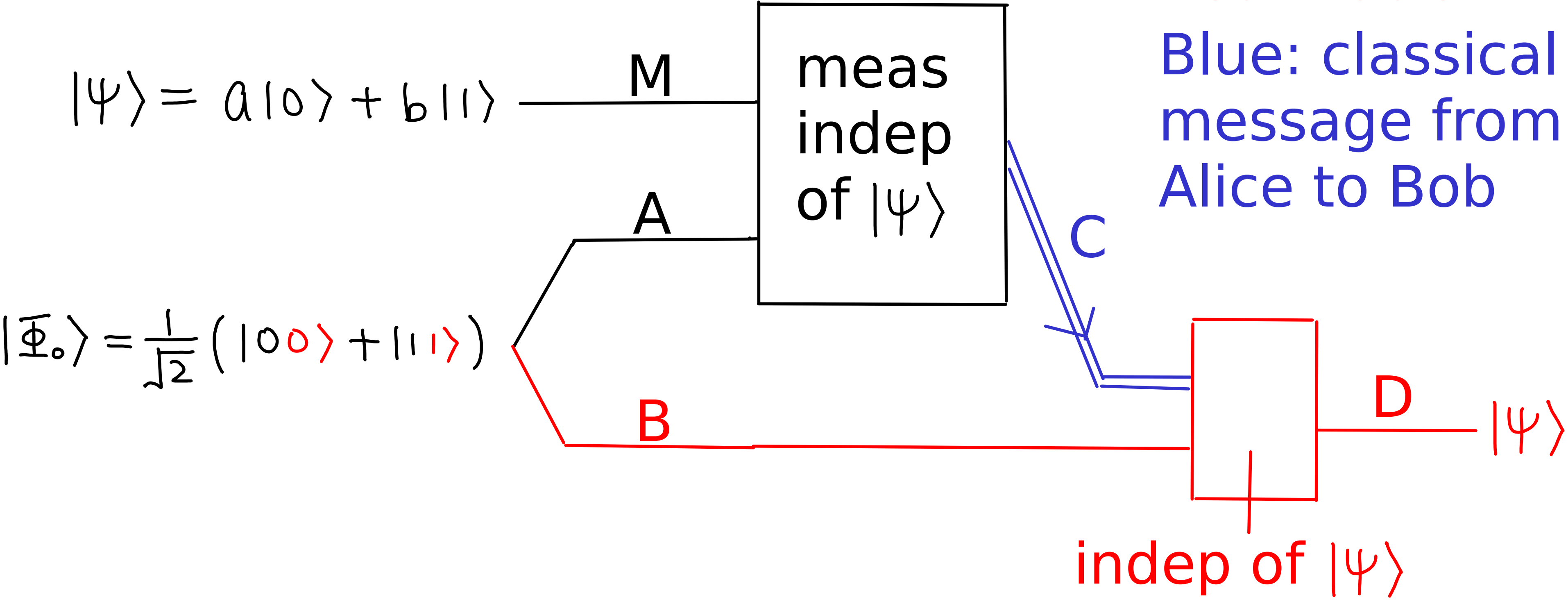
Schematic diagram to be completed:

Black: Alice's
Red: Bob's
Blue: classical message from Alice to Bob



Main mathematical tool:
Expressing an 8-dim quantum state in 2 ways.

$$(a|0\rangle + b|1\rangle)_M \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB}$$

$$= (a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle)_{MAB} \frac{1}{\sqrt{2}}$$

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Pauli's: $\sigma_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

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$|\psi\rangle$ $\qquad\qquad\qquad\qquad$ $|\Phi_0\rangle$

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$|\Phi_0\rangle$ $$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{MA}(a|0\rangle + b|1\rangle)_B \frac{1}{2}$$ $|\psi\rangle$

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$$\overbrace{(a|0\rangle + b|1\rangle)_M}^{|\Psi\rangle} \; \overbrace{\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB}}^{|\Phi_0\rangle} =$$

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If Alice measures MA along the Bell basis, each outcome $k \in \{0,x,y,z\}$ occurs with prob 1/4, and postmeasurement state is $|\Phi_k\rangle_{MA} \otimes \sigma_k |\Psi\rangle_B$.

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If Alice sends k to Bob, he can apply σ_k to B, turning $\sigma_k|\psi\rangle_B$ to $|\psi\rangle_B$.

Teleportation

Alice can communicate a qubit to Bob
if (1) she can send 2 classical bits to Bob, and
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Schematic diagram:

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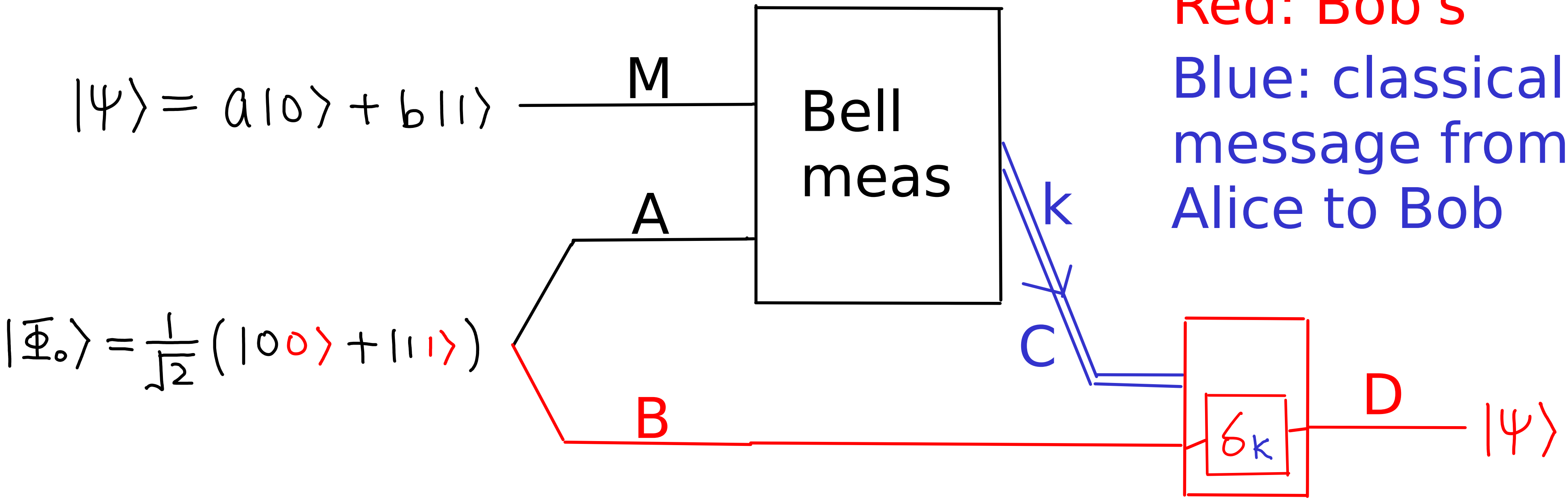

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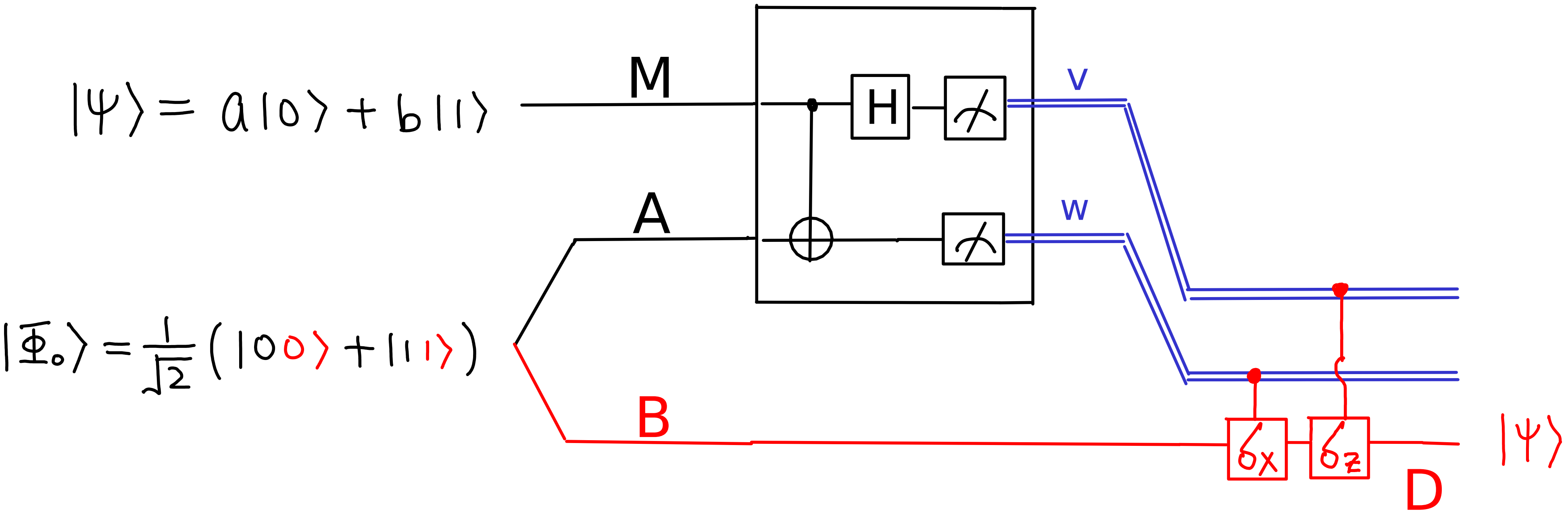
Exercise: verify the following specific implementation



Here, k is given by 2 bits (v,w). Note also $\sigma_y = \sigma_z \cdot \sigma_x$.

General: $|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle$ on RS.

real ortho-normal unit vector on S

For any measurement on S given by projectors $\{P_k\}$

$$I \otimes P_k |\psi\rangle = \sum_i a_i |i\rangle \otimes P_k |\eta_i\rangle$$

pr(k) = $\| I \otimes P_k |\psi\rangle \|^2 = \sum_i a_i^2 \| P_k |\eta_i\rangle \|^2$

$$= \sum_i a_i^2 \, \mathrm{tr}\, P_k |\eta_i\rangle\langle\eta_i| P_k$$

$$= \sum_i a_i^2 \, \mathrm{tr}\, P_k |\eta_i\rangle\langle\eta_i|$$

$$= \mathrm{tr}\, P_k \underbrace{\left(\sum_i a_i^2 |\eta_i\rangle\langle\eta_i| \right)}$$ where $a_i |\eta_i\rangle_S = \langle i| \otimes I \, |\psi\rangle$.

ρ_S : density matrix on S

dxd if d = dim(S)

trace 1, positive semidefinite

Revised formulation of QM:

Revised description of quantum state:

$$|\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \longrightarrow |\psi\rangle\langle\psi| \longrightarrow \sum_i a_i^2 |\eta_i\rangle\langle\eta_i| = \rho_S$$

1. outer product 2. partial trace

revised description of measurement:

$$\mathrm{pr}(k) = \| I \otimes P_k |\psi\rangle \|^2 \longrightarrow \mathrm{pr}(k) = \mathrm{tr}\, P_k \rho_S$$

Define partial trace (describing a state on S from a state on RS) so postmeasurement states & dynamics also makes sense.

The partial trace

Recall the trace of a matrix $\overset{d\times d}{M}$ is the sum of all the diagonal elements. In the Dirac notation:

$$\operatorname{tr} M \underset{\text{insert identity}}{=} \operatorname{tr}\Big(M \sum_{i=1}^{d} \underset{\{|i\rangle\}\ \text{basis}}{|i\rangle\langle i|}\Big) \underset{\text{tr is cyclic and linear}}{=} \sum_{i=1}^{d} \langle i|M|i\rangle$$

Definition: the partial trace of system $\overset{d \text{ dim, basis} \{|i\rangle\}}{B}$, denoted tr_B, is defined on matrices acting on systems $\underset{d_A \text{ dim}}{A}B$ as

$$\operatorname{tr}_B M = \underbrace{\sum_{i=1}^{d} (\underset{A}{I}\otimes\underset{B}{\langle i|})\ \underset{d_A d \times d_A d}{M}\ (I\otimes|i\rangle)}_{d_A \times d_A}$$

The partial trace (example for 2 qubits)

$$I \otimes \langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} [1\ \ 0] & [0\ \ 0] \\ [0\ \ 0] & [1\ \ 0] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$I \otimes \langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} [0\ \ 1] & [0\ \ 0] \\ [0\ \ 0] & [0\ \ 1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(I \otimes \langle 0|)\, M\, (I \otimes |0\rangle) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{bmatrix}$$

$$(I \otimes \langle 1|)\, M\, (I \otimes |1\rangle) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{22} & m_{24} \\ m_{42} & m_{44} \end{bmatrix}$$

$$\mathrm{tr}_B\, M = \sum_{i=1}^{d} (I \otimes \langle i|)\, M\, (I \otimes |i\rangle) = \begin{bmatrix} m_{11} + m_{22} & m_{13} + m_{24} \\ m_{31} + m_{42} & m_{33} + m_{44} \end{bmatrix}$$

$$\mathrm{tr}_B M = \begin{bmatrix} \mathrm{tr}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} & \mathrm{tr}\begin{bmatrix} m_{13} & m_{14} \\ m_{23} & m_{24} \end{bmatrix} \\ \mathrm{tr}\begin{bmatrix} m_{31} & m_{32} \\ m_{41} & m_{42} \end{bmatrix} & \mathrm{tr}\begin{bmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{bmatrix} \end{bmatrix} \xleftarrow{\text{tracing each block}} \left[\begin{array}{cc|cc} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ \hline m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{array}\right] = M$$

summing diagonal blocks

Exercise:

$$\mathrm{tr}_A M = \mathrm{tr}_A \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{bmatrix}$$

Example: A, B are 3- and 2-dim respectively. (M: 6x6)

$$M = \left[\begin{array}{c|c|c} M_{11} & M_{12} & M_{13} \\ \hline M_{21} & M_{22} & M_{23} \\ \hline M_{31} & M_{32} & M_{33} \end{array}\right]$$

Each M_{ij} is a 2x2 matrix.

$$\mathrm{tr}_A M = M_{11} + M_{22} + M_{33}$$

(note, the reduced matrix on B is 2x2)

$$\mathrm{tr}_B M = \left[\begin{array}{c|c|c} \mathrm{tr}\, M_{11} & \mathrm{tr}\, M_{12} & \mathrm{tr}\, M_{13} \\ \hline \mathrm{tr}\, M_{21} & \mathrm{tr}\, M_{22} & \mathrm{tr}\, M_{23} \\ \hline \mathrm{tr}\, M_{31} & \mathrm{tr}\, M_{32} & \mathrm{tr}\, M_{33} \end{array}\right]$$

(note, the reduced matrix on A is 3x3)

Remark:

The trace of an r-dim system is a linear map from r x r matrices to real numbers.

The partial trace of an r-dim system is a linear map from rs x rs matrices to s x s matrices where the trace is applied to R, and the identity map on S.

It acts on tensor product matrices as:

$$\operatorname{tr}_R M_R \otimes M_S = \underbrace{(\operatorname{tr} M_R)}_{\text{scalar}} \underbrace{\cdot}_{\text{scalar product}} M_S$$

and extends to any rs x rs matrix.

What is the most general transformation allowed by QM?

Any reasonable transformation N should take quantum states to quantum states !

Viewing N as a mapping from matrices to matrices:

(1) N is linear (QM is)

(2) N is trace preserving: tr(N(M)) = tr(M)
(conservation of probability when M = ρ)

(3) N is completely positive: $M \geq 0 \Rightarrow I \otimes N(M) \geq 0$

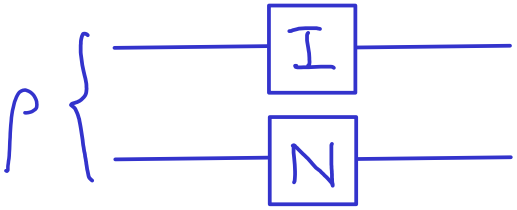
N applied to 1 out of 2 systems takes a valid initial joint state $\rho \geq 0$ to a valid new joint state $I \otimes N(\rho) \geq 0$,

e.g., hold for conjugation by unitaries and partial trace.

The identity map:

Consider the map $I(M) = M$. It is linear, trace preserving and completely positive. It represents the evolution in which nothing happens.

The identity map is most often used when one of two system is being transformed.



On a tensor product input, $I \otimes N(\sigma \otimes \xi) = \sigma \otimes N(\xi)$.

Then, linearity allows the most general $I \otimes N(\rho)$ to be computed.

Definition: a quantum operation is a mapping from matrices to matrices that is linear, trace-preserving, and completely positive.

Synonyms: quantum channel, TCP map ...

Fairly immediate from the definition:

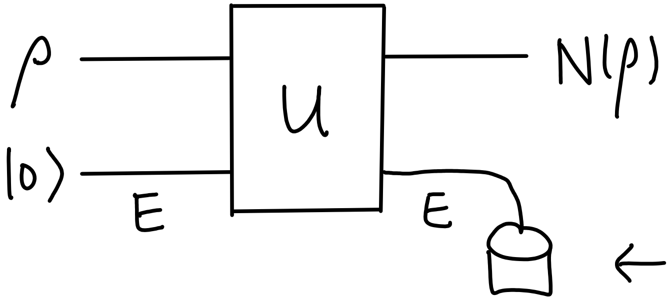
1. Composition of two quantum ops is a quantum op.
   (All 3 properties are preserved by composition.)

2. Tensor product of two quantum ops (applied to two disjoint systems) is a quantum op.

Example 1: Conjugation by unitary $N(\rho) = U \rho U^{\dagger}$

Example 2: Partial trace $N(\rho) = \mathrm{tr}_R \, \rho_{RS}$.

Example 3: $N(\rho) = \mathrm{tr}_E \left(U \rho \otimes |0\rangle\langle 0|_E U^{\dagger} \right)$ is a quantum operation for any system E and any U.



← partial trace

Proof: by examples 1-2 and composition.

Extensions: E can start in any other density matrix uncorrelated with ρ , and partial trace can be taken over a system of any size.

Example: amplitude damping channel

We can define U by its action on a pure qubit state:

$$U\left(a|0\rangle + b|1\rangle\right)_A = a|00\rangle_{EB} + b\left(\sqrt{1-\gamma}\,|01\rangle + \sqrt{\gamma}\,|\underset{\uparrow}{1}0\rangle\right)_{EB}$$

the excitation is transfered from A to E

NB A, B, E all 2-dim.

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$$U = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \\ 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

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$$U = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \\ 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

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NB A, B, E all 2-dim.

On a general density matrix $\rho = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$,

$$U \rho U^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \\ 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & \sqrt{\gamma} & 0 \end{bmatrix} = \begin{pmatrix} c & \sqrt{1-\gamma}\,d & \sqrt{\gamma}\,d & 0 \\ \sqrt{1-\gamma}\,e & (1-\gamma)f & \sqrt{\gamma}\sqrt{1-\gamma}\,f & 0 \\ \sqrt{\gamma}\,e & \sqrt{\gamma}\sqrt{1-\gamma}\,f & \gamma f & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{tr}_E\, U \rho U^{\dagger} = \mathrm{tr}_E \begin{pmatrix} c & \sqrt{1-\gamma}\, d & \sqrt{\gamma}\, d & 0 \\ \sqrt{1-\gamma}\, e & (1-\gamma) f & \sqrt{\gamma}\sqrt{1-\gamma}\, f & 0 \\ \sqrt{\gamma}\, e & \sqrt{\gamma}\sqrt{1-\gamma}\, f & \gamma f & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} c & \sqrt{1-\gamma}\, d \\ \sqrt{1-\gamma}\, e & (1-\gamma) f \end{bmatrix} + \begin{bmatrix} \gamma f & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c + \gamma f & \sqrt{1-\gamma}\, d \\ \sqrt{1-\gamma}\, e & (1-\gamma) f \end{bmatrix}$$

$$\operatorname{tr}_E U \rho U^{\dagger} = \operatorname{tr}_E \begin{pmatrix} c & \sqrt{1-\gamma}\, d & \sqrt{\gamma}\, d & 0 \\ \sqrt{1-\gamma}\, e & (1-\gamma) f & \sqrt{\gamma}\sqrt{1-\gamma}\, f & 0 \\ \sqrt{\gamma}\, e & \sqrt{\gamma}\sqrt{1-\gamma}\, f & \gamma f & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} c & \sqrt{1-\gamma}\, d \\ \sqrt{1-\gamma}\, e & (1-\gamma) f \end{bmatrix} + \begin{bmatrix} \gamma f & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c+\gamma f & \sqrt{1-\gamma}\, d \\ \sqrt{1-\gamma}\, e & (1-\gamma) f \end{bmatrix}$$

So, the channel takes $\rho = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$ to $\begin{bmatrix} c+\gamma f & \sqrt{1-\gamma}\, d \\ \sqrt{1-\gamma}\, e & (1-\gamma) f \end{bmatrix}$

A fraction γ of the (1,1) entry is moved to the (0,0) entry, and the off diagonal terms are diminished.

What is $N(\rho)$ in terms of U?

Let $$U = \sum_{j=0}^{d_E-1}\sum_{k=0}^{d_E-1} |j\rangle\langle k|_E \otimes U_{jk} = \left[\begin{array}{c|c|c} U_{00} & U_{01} & U_{02} \cdots \\ \hline U_{10} & U_{11} & U_{12} \cdots \\ \hline U_{20} & U_{21} & U_{22} \\ \vdots & \vdots & \ddots \end{array}\right]$$

E: 1st register.

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E: 1st register.

$$N(\rho) = \mathrm{tr}_E\left(U \rho \otimes |0\rangle\langle 0|_E U^{\dagger} \right)$$

$$= \mathrm{tr}_E\left(\sum_{j=0}^{d_E-1}\sum_{k=0}^{d_E-1} |j\rangle\langle k|_E \otimes U_{jk} \right)\left(|0\rangle\langle 0|_E \otimes \rho \right)\left(\sum_{j'=0}^{d_E-1}\sum_{k'=0}^{d_E-1} |k'\rangle\langle j|_E \otimes U^{\dagger}_{j'k'} \right)$$

What is $N(\rho)$ in terms of U?

isometry

Let $U = \sum_{j=0}^{d_E-1}\sum_{k=0}^{d_E-1} |j\rangle\langle k|_E \otimes U_{jk} = \begin{pmatrix} U_{00} & U_{01} & U_{02} & \cdots \\ U_{10} & U_{11} & U_{12} & \cdots \\ U_{20} & U_{21} & U_{22} & \\ \vdots & \vdots & & \ddots \end{pmatrix}$

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$$= \mathrm{tr}_E\Big(\underbrace{\sum_{j=0}^{d_E-1} |j\rangle_E \otimes U_{j0}}_{\text{isometry}}\Big)\Big(\underset{\substack{\uparrow \\ \text{1-dim}}}{1} \otimes \rho\Big)\Big(\sum_{j'=0}^{d_E-1} \langle j'|_E \otimes U^{\dagger}_{j'0}\Big)$$

can be omitted

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Let $$U = \sum_{j=0}^{d_E-1}\sum_{k=0}^{d_E-1} |j\rangle\langle k|_E \otimes U_{jk} = \begin{array}{c} \text{isometry} \\ \left(\begin{array}{c|c|c} U_{00} & U_{01} & U_{02} \cdots \\ \hline U_{10} & U_{11} & U_{12} \cdots \\ \hline U_{20} & U_{21} & U_{22} \\ \vdots & \vdots & \ddots \end{array}\right) \end{array}$$

E: 1st register.

$$N(\rho) = \mathrm{tr}_E\left(U\, \rho \otimes |0\rangle\langle 0|_E\, U^\dagger\right)$$

$$= \mathrm{tr}_E\left(\sum_{j=0}^{d_E-1}\sum_{k=0}^{d_E-1} |j\rangle\langle k|_E \otimes U_{jk}\right)\left(|0\rangle\langle 0|_E \otimes \rho\right)\left(\sum_{j'=0}^{d_E-1}\sum_{k'=0}^{d_E-1} |k'\rangle\langle j'|_E \otimes U^\dagger_{j'k'}\right)$$

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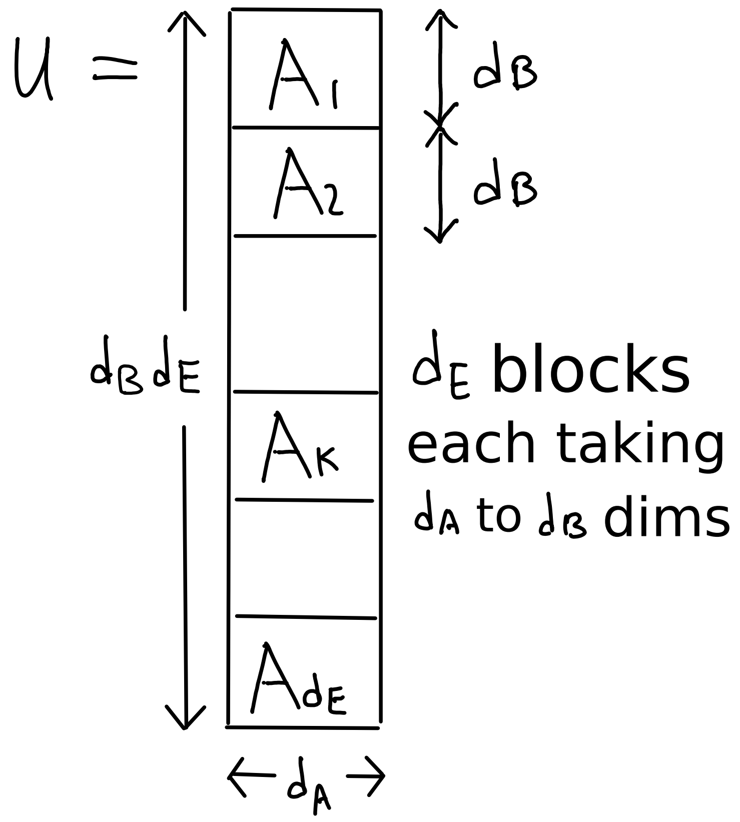
can be omitted

$$= \sum_{j=0}^{d_E-1} U_{j0}\, \rho\, U_{j0}^\dagger \quad \text{mixture of states } \frac{U_{j0}\,\rho\, U_{j0}^\dagger}{\mathrm{tr}\, U_{j0}^\dagger U_{j0}\, \rho} \ \text{wp } \mathrm{tr}\, U_{j0}^\dagger U_{j0}\, \rho$$

not nec unitary

More generally, let U be an isometry taking system A to system BE (dims of A, B, and E are arbitrary).



$$U = \sum_{k=1}^{d_E} |k\rangle_E \otimes A_k$$

Stinespring dilation,
isometric extension

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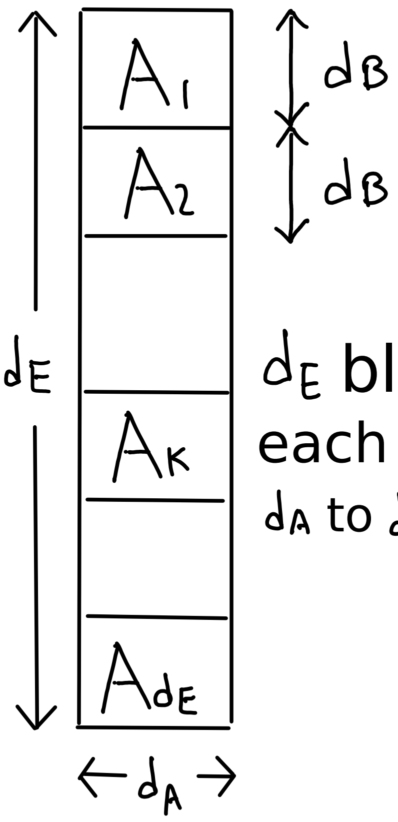
$U =$



d_E blocks each taking d_A to d_B dims

$$N(\rho) = tr_E(U \rho U^\dagger) = \sum_{k=1}^{d_E} A_k \rho A_k^\dagger$$

Kraus representation of N

A_k's : Kraus operators

not $A_k^\dagger$'s

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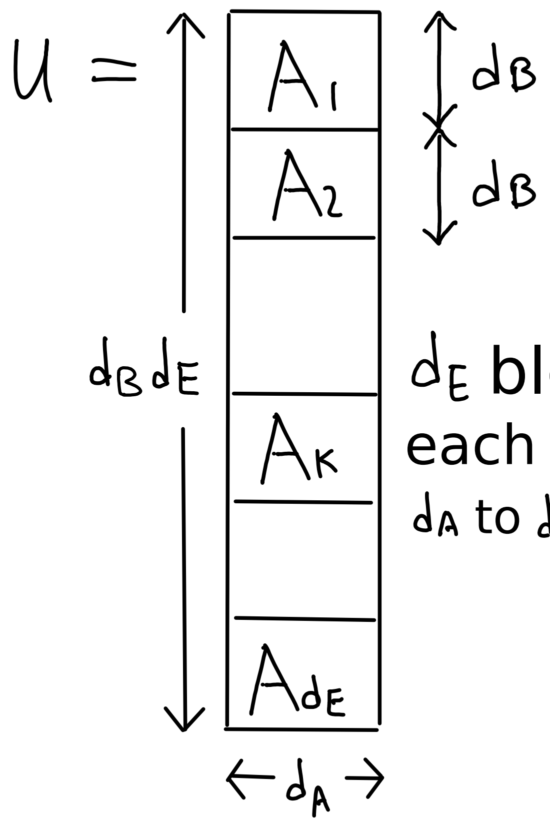

d_E blocks each taking d_A to d_B dims

$$U = \sum_{k=1}^{d_E} |k\rangle_E \otimes A_k$$

Stinespring dilation, isometric extension

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* A map w/ Kraus representation is linear and completely positive

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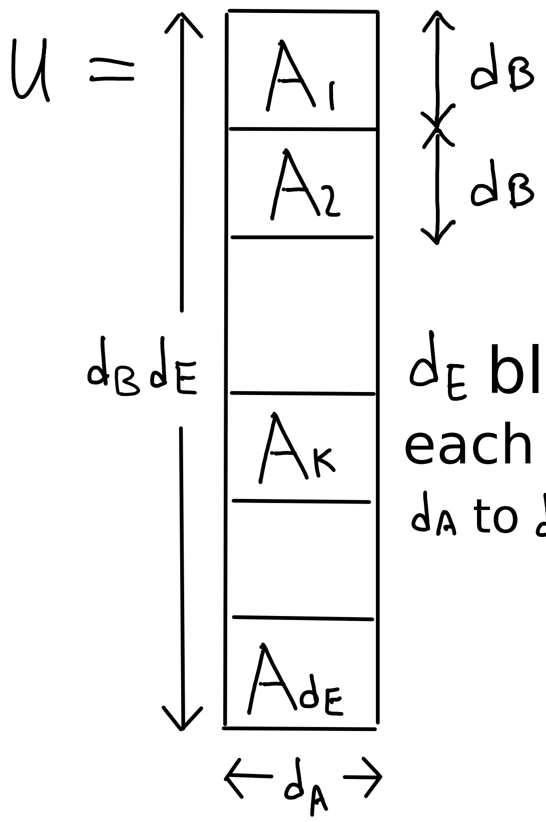

d_E blocks each taking d_A to d_B dims

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Kraus representation of N

A_k's : Kraus operators

* A map w/ Kraus representation is linear and completely positive

* U isometry $\Leftrightarrow U^\dagger U = I_A$

$$\Leftrightarrow \sum_{k=1}^{d_E} A_k^\dagger A_k = I_A$$

$\Leftrightarrow N$ trace preserving

$$U = \sum_{k=1}^{d_E} |k\rangle_E \otimes A_k$$

Stinespring dilation, isometric extension

Example: amplitude damping channel

$$U = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \\ \hline 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$$

$$N(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger$$

Ex: check $A_0^\dagger A_0 + A_1^\dagger A_1 = I$

Example: amplitude damping channel

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If the initial state is $|\psi\rangle = a|0\rangle + b|1\rangle$ $\left(\rho = |\psi\rangle\langle\psi|\right)$
output is the mixture of two unnormalized states:

$$A_0|\psi\rangle = a|0\rangle + \sqrt{1-\gamma}\, b|1\rangle$$

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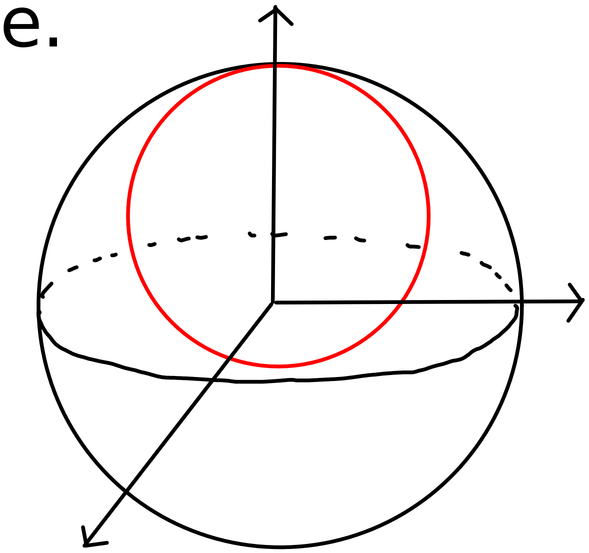
Interpretation: $|0\rangle$: ground state

$|1\rangle$: excited state

A_1 : de-excitation (with prob γ)

A_0 : no de-excitation, but diminished amplitude for $|1\rangle$

Execise: evaluate $N(\frac{1}{2}I + aX + bY + cZ)$ and find how N transform the Bloch sphere.



The ground state $|0\rangle\langle 0|$ is a fixed point of N.
N is not unital (taking the identity matrix to itself).

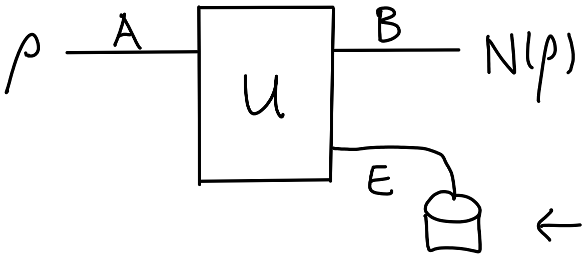
Theorem: any quantum operation N from system A to system B can be represented as $N(\rho) = \mathrm{tr}_E(U \rho U^\dagger)$ for some system E and some Stinespring dilation U.

Proof omitted. See arxiv.org/abs/quant-ph/0201119

Representations of quantum operations:

✓ 1. Unitary representation

$$N(\rho) = \mathrm{tr}_E (U \rho U^\dagger)$$



← partial trace

✓ 2a. Kraus rep: $N(\rho) = \sum_{k=1}^{d_E} A_k \rho A_k^\dagger \, , \; \sum_{k=1}^{d_E} A_k^\dagger A_k = I_A$

2b. Conversely, given d_E operators A_k mapping from system A to B satisfying $\sum_{k=1}^{d_E} A_k^\dagger A_k = I_A$, $U = \sum_{k=1}^{d_E} |k\rangle_E \otimes A_k$ is an isometry, and $\mathrm{tr}_E (U \rho U^\dagger) = \sum_{k=1}^{d_E} A_k \rho A_k^\dagger$

3. $N(\rho)$ as an explicit function of ρ e.g. $\begin{bmatrix} c & d \\ e & f \end{bmatrix} \rightarrow \begin{bmatrix} c+\gamma f & \sqrt{1-\gamma}\, d \\ \sqrt{1\gamma}\, e & (1-\gamma) f \end{bmatrix}$

4. Choi matrix (reading)