2020-09-14 self-study notes

1. Locality of quantum mechanics

2. Teleportation and superdense coding for
   arbitrary dimension, and the generalized Pauli basis

Locality of quantum mechanics

Suppose Alice and Bob each holds one quantum system, and they share a joint initial state.

If Alice measures her system A, the GLOBAL state (and Bob's state on B) post-measurement may depend her measurement outcome.

e.g., sharing $\frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$

Alice measures along the $\{|0\rangle, |1\rangle\}$ basis.

If her outcome is "0" Bob's state is $|0\rangle$.

If her outcome is "1" Bob's state is $|1\rangle$.

Question: can Alice signal to Bob (transmitting a message) by measuring her system?

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Better not!

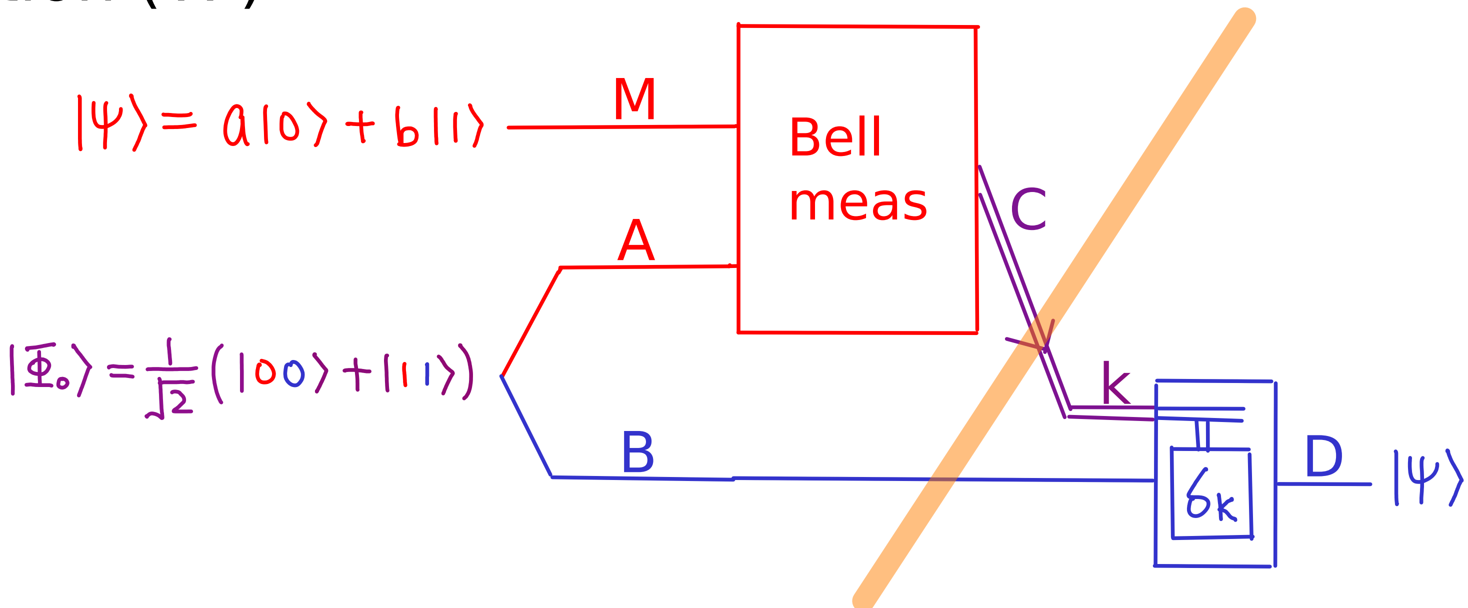
1. Alice cannot control the outcome, so, even though Bob can find out Alice's measurement outcome, the net result is the sharing of a random bit.

   Resource inequality: ebit $\geqslant$ rbit

2. Bob doesn't even know if Alice has made the meas or if she ever would. Any measurement Bob can perform depends on his local state, WHETHER Alice has measured or not. Bob's state is $\frac{I}{2}$.

Locality of teleportation before the classical message:

Teleportation (TP)



Consider the orange "time slice" -- after Alice's Bell measurement but before the outcome k reaches Bob.

For each k, what is Bob's state (at the orange time)?

Ans: $\sigma_k|\psi\rangle$

From Alice's, conditioned on the outcome k, sees the global state as:

$$|k\rangle_C \otimes \left(\sigma_k |\psi\rangle\right)_B$$

(the above is a pure state, so, we can write either the vector or the density matrix)

For Bob, he doesn't know k yet. So, his state is mixed, and the density matrix is given by:

$$\sum_k \Pr(k)\ \sigma_k |\psi\rangle\langle\psi| \sigma_k^\dagger = \frac{1}{4} \sum_k \sigma_k |\psi\rangle\langle\psi| \sigma_k^\dagger$$

(since the 4 outcomes are equiprobable)

Define the following quantum channel mapping from 2x2 matrices to 2x2 matrices:

$$R(\rho) = \frac{1}{4}\left(\rho + \sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z\right)$$

Show that for all hermitian ρ

$$R(\rho) = \mathrm{tr}(\rho) \cdot \frac{I}{2}$$

One simple way to show this is to write any 2x2 matrices as a real linear combination

$$\rho = a I + b \sigma_x + c \sigma_y + d \sigma_z$$

and note that for each of $\sigma_x, \sigma_y, \sigma_z$

commutes with 2 of $I, \sigma_x, \sigma_y, \sigma_z$ and anticommutes

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e.g., for σ_x :

$I \sigma_x I = \sigma_x$

$\sigma_x \sigma_x \sigma_x = \sigma_x$

$\sigma_y \sigma_x \sigma_y = -\sigma_x$

$\sigma_z \sigma_x \sigma_z = -\sigma_x$

σ_x commutes with I, σ_x

σ_x anticommutes with σ_y, σ_z

So $R(\sigma_x) = \frac{1}{4}(\sigma_x + \sigma_x \sigma_x \sigma_x + \sigma_y \sigma_x \sigma_y + \sigma_z \sigma_x \sigma_z)$

$+\sigma_x$ $\qquad\qquad$ $-\sigma_x$

$\therefore R(\sigma_x) = 0$

Similarly, $R(\sigma_y) = R(\sigma_z) = 0$

Meanwhile, $R(I) = I$, So, for $\rho = aI + b\sigma_x + c\sigma_y + d\sigma_z$

$R(\rho) = aI = \frac{\mathrm{tr}(\rho)}{2} \times I$.

$\mathcal{R}$ is called the completely randomizing channel.

Any input state is transformed to I/2 (the completely mixed state).

Back to teleportation:
Bob's state at the orange time is $\frac{1}{4}\sum_k \sigma_k |\psi\rangle\langle\psi| \sigma_k^+ = \frac{I}{2}$

This is consistent with a second derivation of his state at the orange time: he holds qubit B of the joint state

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)_{AB} = |\Phi_0\rangle$$

So, his state is $tr_A |\Phi_0\rangle\langle\Phi_0| = \frac{I}{2}$

(if doing this partial trace is not familiar to you, let me know)

So, Bob's state is I/2 UNTIL the message k suddenly arrives and his state changes suddenly to $\sigma_k |\psi\rangle$.

Theorem: superdense coding (Bennett-Wiesner 93)

Suppose Alice and Bob share the state $\frac{1}{\sqrt{s}}\sum_{i=1}^{s}|i\rangle\otimes|i\rangle$ and Alice can send an s-dimensional quantum system to Bob. Then, Alice can communicate $t=s^2$ messages to Bob!

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Exercise:

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Note that for $s = 2^{2n}$,

we can simply repeat "SD coding for s=2" n times.

For general s: let ω be a primitive s-th root of unity.

Let Alice's message be $(k,j) \in \{1,2,\ldots,s\} \times \{1,2,\ldots,s\}$.

Consider the s^2 unitaries:

$$U_{kj} = \underbrace{\sum_i |i+k \bmod s\rangle\langle i|}_{\text{like } \sigma_x^k} \cdot \underbrace{\sum_l \omega^{jl} |l\rangle\langle l|}_{\text{like } \sigma_z^j}$$

NB for s=2, $\sigma_y = \sigma_x \sigma_z$. U_{kj}'s are "generalized Pauli's."

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$$= \sum_i \omega^{ji} |i+k \bmod S\rangle\langle i|$$

Verify that the S^2 states

$$|\Phi_{kj}\rangle = U_{kj} \frac{1}{\sqrt{S}} \sum_{u=1}^{S} |u\rangle \otimes |u\rangle$$ are mutually orthogonal.

Similarly, for teleportation of an s-dim system :

Alice, instead of the Bell measurement, should measure along the basis: $\{|\Phi_{kj}\rangle\}$

Exercise: if Alice's input state is $|\psi\rangle \in \mathbb{C}^s$
and the measurement outcome is (k,j)
what is the postmeasuement state on Bob's half of the maximally entangled state?