2020-09-21 self-study notes

1. Absorbing operations into measurements
   (with possible dim change)

2. Transpose trick with possible dim change

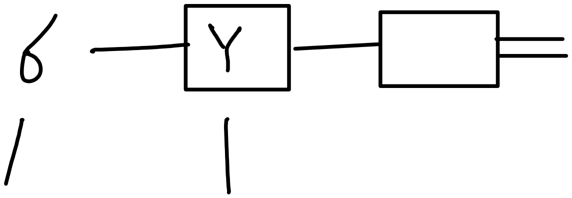
3. A useful lemma on partial tracing

4. Reference to monotonicity of trace distance
   under quantum operations
   (Watrous book, p34, (1.182)-(1.183), and
   parts of p32-33 may also help.)

1. Absorbing operations into measurements (with possible dim change)

Lemma: $\mathrm{tr}\; Y \rho Y^{\dagger} H = \mathrm{tr}\; \rho\, Y^{\dagger} H Y$

Interpretation:



H: any of the meas operators

state

transformation

not necessarily unitary
or even dim preserving

eg $\rho : d \times d$, $H : d' \times d'$, $Y : \mathbb{C}^{d} \to \mathbb{C}^{d'}$

NB: on LHS, $Y\rho Y^{\dagger}$, $H : d' \times d'$, on RHS, $Y^{\dagger} H Y$, $\rho : d \times d$

Special & familiar case: Y unitary, meas projective.
A unitary before a meas is equivalent to changing
the measurement basis the opposite way.

Proof of lemma: use linearity on σ & H. (but not on Y !!)

suffices to show for $\sigma = |i\rangle\langle j|$, $H = |k\rangle\langle \ell|$

$i, j \in \{1, \dots, d\}$, $k, \ell \in \{1, \dots, d'\}$

$$\begin{aligned}
\text{LHS} &= \operatorname{tr} Y \sigma Y^\dagger H \\
&= \operatorname{tr} Y |i\rangle\langle j| Y^\dagger |k\rangle\langle \ell| \\
&= \langle \ell | Y | i\rangle \langle j | Y^\dagger | k\rangle
\end{aligned}$$

apply def of tr

$$\begin{aligned}
\text{RHS} &= \operatorname{tr} \sigma Y^\dagger H Y \\
&= \operatorname{tr} |i\rangle\langle j| Y^\dagger |k\rangle\langle \ell| Y \\
&= \langle j | Y^\dagger | k\rangle \langle \ell | Y | i\rangle = \text{LHS}
\end{aligned}$$

Question: when we absorb operations into meas operators, do we get the correct postmeas state ? Why ?

Answer: yes.

Idea: if the replacement (of operation + meas by a new measurement) is exact (as quantum operations) then, the replacement, composed with the identity operation on the rest of the universe, is still exact.

This statement holds independent of the input, which can be arbitrarily correlated between the measured system and the rest of the universe containing the postmeas state. So, the postmeas state must also be correct.

2. Transpose trick with possible dim change

Lemma: Let $X : \mathbb{C}^d \to \mathbb{C}^{d'}$. Then

$$I \otimes X \sum_{i=1}^{d} |i\rangle |i\rangle = X^T \otimes I \sum_{j=1}^{d'} |j\rangle |j\rangle$$

Proof: write $X = \sum_{k=1}^{d'} \sum_{\ell=1}^{d} x_{k,\ell} |k\rangle\langle \ell|$

$$\text{LHS} = I \otimes X \sum_{i=1}^{d} |i\rangle|i\rangle = \sum_{i=1}^{d} |i\rangle \otimes \left(X |i\rangle \right)$$

$$= \sum_{i=1}^{d} |i\rangle \otimes \sum_{k=1}^{d'} \sum_{\ell=1}^{d} x_{k,\ell} |k\rangle \underbrace{\langle \ell | i\rangle}_{\delta_{i\ell}}$$

$$= \sum_{k=1}^{d'} \sum_{\ell=1}^{d} x_{k,\ell} |\ell\rangle \otimes |k\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'}$$

$$X^T = \sum_{k=1}^{d'} \sum_{l=1}^{d} x_{k,l} \, |l\rangle\langle k|$$

$$\text{RHS} = X^T \otimes I \sum_{j=1}^{d'} |j\rangle |j\rangle = \sum_{j=1}^{d'} \left(X^T |j\rangle \right) \otimes |j\rangle$$

$$= \sum_{j=1}^{d'} \sum_{k=1}^{d'} \sum_{l=1}^{d} x_{k,l} \, |l\rangle \underbrace{\langle k|j\rangle}_{\delta_{kj}} \otimes |j\rangle$$

$$= \sum_{k=1}^{d'} \sum_{l=1}^{d} x_{k,l} \, |l\rangle \otimes |k\rangle = \text{LHS} .$$

3. A useful lemma on partial tracing

Recall: $M : d_1 d_2 \times d_1 d_2$, $\{|i\rangle\}_{i=1}^{d_1}$ basis for $\mathbb{C}^{d_1}$

$$\operatorname{tr}_1 M = \sum_{i=1} \left(\langle i| \otimes I \right) M \left(|i\rangle \otimes I \right)$$

Also partial trace is indep of basis used in the tracing.

Lemma 1: Let ρ be a $d \times d$ matrix, $|\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle|i\rangle$

Then $\mathrm{tr}_1 (\rho \otimes I) |\Phi_d\rangle\langle\Phi_d| \cdot d = \rho^T$.

Proof: $\mathrm{tr}_1 (\rho \otimes I) |\Phi_d\rangle\langle\Phi_d| \, d$

$$= \sum_{k=1}^{d} (\langle k| \otimes I) \cdot \Big((\rho \otimes I) \underbrace{|\Phi_d\rangle\langle\Phi_d| \, d} \Big) \cdot (|k\rangle \otimes I)$$

$$= \sum_{k=1}^{d} (\langle k| \otimes I) \cdot \Big((\rho \otimes I) \sum_{i=1}^{d} |i\rangle|i\rangle \sum_{j=1}^{d} \langle j|\langle j| \Big) \cdot (|k\rangle \otimes I)$$

color ops in 1st sys red, and those in 2nd sys green

$$= \sum_{k=1}^{d} (\langle k| \otimes I) \cdot \Big((\rho \otimes I) \sum_{i=1}^{d} |i\rangle|i\rangle \sum_{j=1}^{d} \langle j|\langle j| \Big) \cdot (|k\rangle \otimes I)$$

multiply ops in each sys

$$= \sum_{k=1}^{d} \sum_{i=1}^{d} \sum_{j=1}^{d} \langle k| \rho |i\rangle \underbrace{\langle j|k\rangle}_{\delta_{jk}} \otimes |i\rangle\langle j| \qquad \text{(1st sys now 1-dim)}$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} \langle j| \rho |i\rangle \otimes |i\rangle\langle j| = \rho^T$$