CO781 / QIC 890:

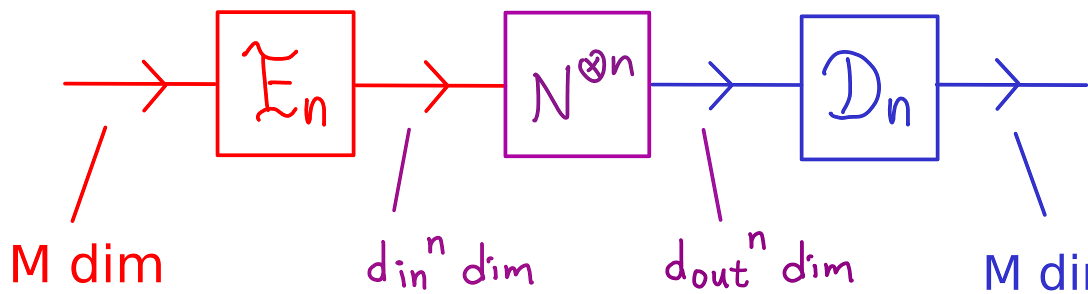
Theory of Quantum Communication

Topics 5, part 1

Transmitting quantum data through a quantum channels

- the various inequivalent error definitions
- the coherent information

Most general (M,n)-code for Q comm through quantum channel



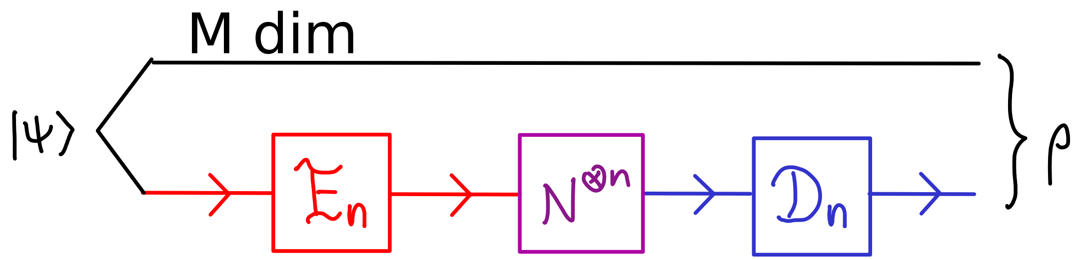
Def: given a quantum channel N, R>0 is an achievable rate if $\forall n$, $\exists$ (M,n)-code transmitting M-dim quantum states with n uses of N, $M \geq 2^{n(R-\delta_n)}$, error ϵ_n, s.t. $\delta_n, \epsilon_n \to 0$.

Def: quantum capacity Q(N) = supremum over achievable rates

How to define ϵ_n ?

Plausible error definitions:

(1)

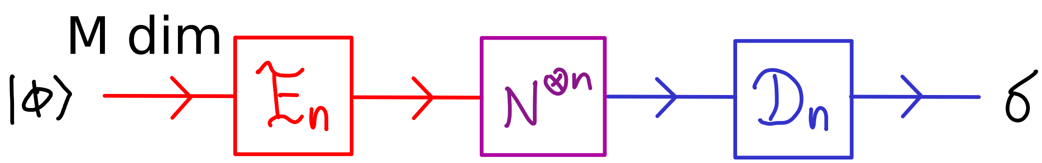


ⓐ $\forall\, |\psi\rangle,\ \langle\psi|\rho|\psi\rangle \geq 1-\epsilon_n$

ie $\| \mathcal{D}_n \circ N^{\otimes n} \circ \mathcal{E}_n - I \|_\diamond \leq f(\epsilon_n)$ — the code simulates identity channel in diamond norm

ⓑ for max ent $|\psi\rangle,\ \langle\psi|\rho|\psi\rangle \geq 1-\epsilon_n$ — Choi-state for the code simulates max entangled state

(2)



ⓐ $\forall\, |\phi\rangle,\ \langle\phi|\sigma|\phi\rangle \geq 1-\epsilon_n$

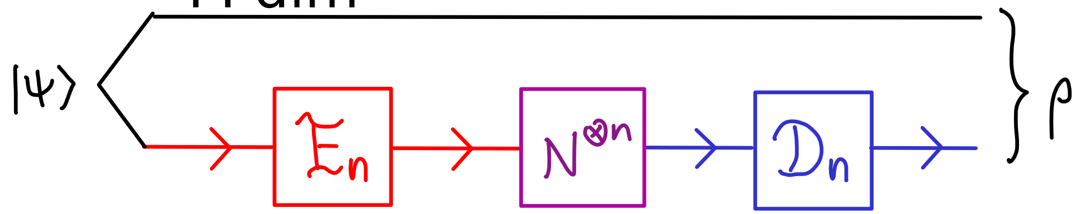
or $\min_{|\phi\rangle} \langle\phi|\sigma|\phi\rangle \geq 1-\epsilon_n$ — worse case pure-input / output overlap

ⓑ $\mathbb{E}_{|\phi\rangle} \langle\phi|\sigma|\phi\rangle \geq 1-\epsilon_n$ — average case pure-input / output overlap

Plausible error definitions:

the defs are not equivalent

(1) M dim



ⓐ $\forall |\psi\rangle, \langle\psi|\rho|\psi\rangle \geq 1-\epsilon_n$

ie $\|\mathcal{D}_n \circ \mathcal{N}^{\otimes n} \circ \mathcal{E}_n - \mathcal{I}\|_\diamond \leq f(\epsilon_n)$

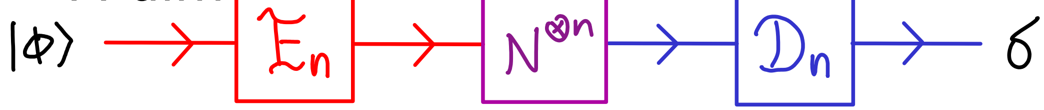
ⓑ for max ent $|\psi\rangle$, $\langle\psi|\rho|\psi\rangle \geq 1-\epsilon_n$

(1) (a) strongest

$\not\Uparrow$

(1) (b) (e.g., one input randomized)

(2) M dim



ⓐ $\forall |\phi\rangle, \langle\phi|\sigma|\phi\rangle \geq 1-\epsilon_n$

or $\min_{|\phi\rangle} \langle\phi|\sigma|\phi\rangle \geq 1-\epsilon_n$ worse case pure-input

ⓑ $\mathbb{E}_{|\phi\rangle} \langle\phi|\sigma|\phi\rangle \geq 1-\epsilon_n$ average case pure-input

(2)(a) w/ $\epsilon_n \implies$ (1)(a) w/ $\frac{3}{2}\epsilon_n$

$\not\Uparrow$

(2)(b)

But all 4 error definitions give the SAME Q(N) !!

9809010 (Barnum, Knill, Nielsen), 0311037 (Kretschmann, (R) Werner)

Idea similar to expunging the worse half of the codewords in the classical case -- here take a good subspace with reduced dim.

Very convenient -- use weak definition when proving a rate is achievable, use stronge definition when using the protocol.

Recall:

$$C(N) = \sup_r \chi^{(r)}(N)$$

r-shot Holevo info for N: an optimization involving input ensemble for r uses of N (and entropic functions)

Will see:

$$Q(N) = \sup_r Q^{(r)}(N)$$

r-shot coherent info for N: an optimization involving input for r uses of N (and entropy functions)

Def: Coherent info for a quantum state

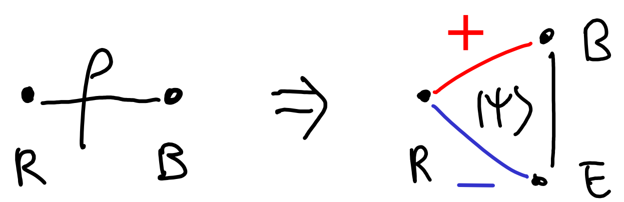
Let ρ be a state on RB.

$$I_c(R\rangle B)_\rho := [S(B) - S(RB)]_\rho$$

The above is the coherent info from R to B. Note R, B not symmetric.

What the coherent info represents (clearer in Church of Larger HS):

Let $|\psi\rangle_{RBE}$ purify ρ_{RB}.



$|\psi\rangle$ unique up to unitary on E

(does not affect any entropy we will calculate)

$$I_c(R\rangle B)_\rho$$

$$:= [S(B) - S(RB)]_\rho$$

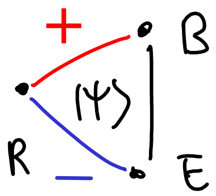
$$= [S(B) - S(E)]_{|\psi\rangle\langle\psi|}$$

antisym wrt BE

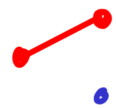
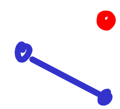
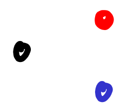
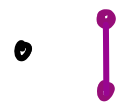
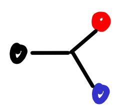
$$= \frac{1}{2}\big[\underbrace{S(B:R)}_{S(B)+\cancel{S(R)}-S(BR)} - \underbrace{S(E:R)}_{S(E)+\cancel{S(R)}-S(ER)}\big]_{|\psi\rangle\langle\psi|}$$

coh info = 1/2 * (how much more R is corr with B than with E in QMI)

Examples:



$|\Phi\rangle = 1$ ebit

	$\|\Psi\rangle_{RBE}$	$S(B)$	$S(E)$	$I_c(R\rangle B)$	$S(R:B)$	$S(R:E)$
	$\|\Phi\rangle_{RB}\|0\rangle_E$	1	0	1	2	0
	$\|0\rangle_B \|\Phi\rangle_{RE}$	0	1	−1	0	2
	$\|0\rangle_R\|0\rangle_B\|0\rangle_E$	0	0	0	0	0
	$\|0\rangle_R \|\Phi\rangle_{BE}$	1	1	0	0	0
	$\|0\rangle\|0\rangle\|0\rangle + \|1\rangle\|1\rangle\|1\rangle_{RBE}$	1	1	0	1	1

quantum correlation between RB

total correlation between RB

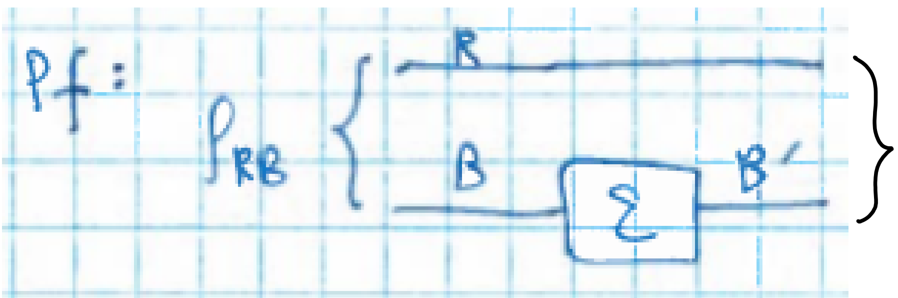
Properties of coherent information Ic(R>B) := S(B) - S(RB)

1. Invariant under local unitaries (on R, B, or E separately)

2. Invariant under attaching / discarding LOCAL PURE states.

3. Non-increasing under local operations (TCP maps) on B.

Pf:



$[I \otimes \mathcal{E}(\rho)]_{RB'}$

Ic(R>B) = S(B) - S(RB) = S(B:R) - S(R)

Moving from ρ to $I \otimes \mathcal{E}(\rho)$:

By monotonicity of QMI, $S(B:R)_\rho \geq S(B':R)_{I \otimes \mathcal{E}(\rho)}$

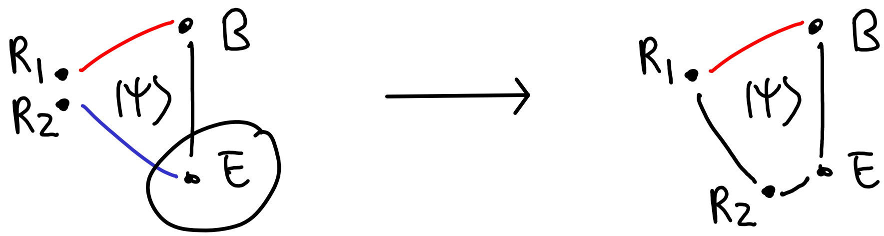
Meanwhile, ρ_R & $S(R)$ unchanged.

$\therefore I_c(R>B)_\rho \geq I_c(R>B')_{I \otimes \mathcal{E}(\rho)}$

Properties of coherent information Ic(R>B) := S(B) - S(RB)

4. Can be increasing or decreasing under local operation of R
in particular, under discarding of a subsystem of R.

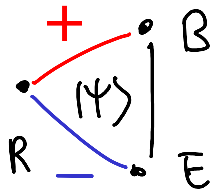
Let R = R1 R2. Discarding R2 means sending R2 to E.



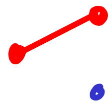
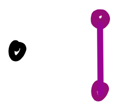
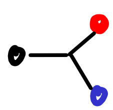
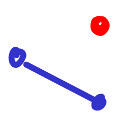
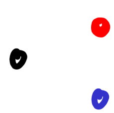
E purifies $\rho_{R_1R_2B}$ R2 E purifies ρ_{R_1B}

Conceptually, the "environment" contains everything not in the 2 systems for which we calculate the coherent info ... so anything discarded goes to the environment.

Examples:



$|\Phi\rangle$ = 1 ebit

	$\|\Psi\rangle_{RBE}$	$S(B)$	$S(E)$	$I_c(R\rangle B)$	$S(R:B)$	$S(R:E)$
	$\|\Phi\rangle_{RB}\|0\rangle_E$	1	0	1	2	0
↓	discard R (and +/- |0>'s), Ic decreases					
	$\|0\rangle_R\ \|\Phi\rangle_{BE}$	1	1	0	0	0
↑	discard R (and +/- |0>'s), Ic unchanged					
	$\|0\rangle\|0\rangle\|0\rangle + \|1\rangle\|1\rangle\|1\rangle_{RBE}$	1	1	0	1	1
	$\|0\rangle_B\ \|\Phi\rangle_{RE}$	0	1	−1	0	2
↓	discard R (and +/- |0>'s), Ic increases					
	$\|0\rangle_R\|0\rangle_B\|0\rangle_E$	0	0	0	0	0

Properties of coherent information Ic(R>B) := S(B) - S(RB)

⑤ Continuity

Recall the Fannes-Alicki ineq

Let $\rho, \sigma \in B(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$, $\|\rho - \sigma\|_1 \leq \delta$

Then $\left| S(A|B)_\rho - S(A|B)_\sigma \right| \leq 4\delta \log d_A + 2h(\delta)$

binary entropy function

Since $I_c(R>B) = S(B) - S(RB) = -S(R|B)$

$\therefore \left| I_c(R>B)_\rho - I_c(R>B)_\sigma \right| \leq 4\delta \log R + 2h(\delta)$

Useful both for the continuity of Q(N) and for proving the converse.

Properties of coherent information Ic(R>B) := S(B) - S(RB)

6. If B = B1 X, where X is a classical system

then, $\rho_{RB} = \rho_{RB_1X} = \sum_x p(x)\, |x\rangle\langle x|_X \otimes \sigma_{x\,RB_1}$

$\rho_B = \rho_{B_1X} = \sum_x p(x)\, |x\rangle\langle x|_X \otimes \sigma_{x\,B_1} \leftarrow \mathrm{tr}_R\, \sigma_{x\,RB_1}$

$$S_{RB} = H(X) + \sum_x p(x)\, S(\sigma_{x\,RB_1})$$

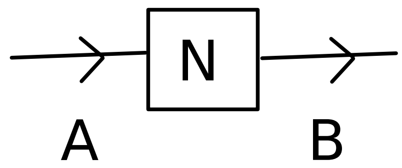
$$S_B = H(X) + \sum_x p(x)\, S(\sigma_{x\,B_1})$$

$$\therefore\ I_c(R\rangle B)_\rho = S_B - S_{RB} = \sum_x p(x) \underbrace{\left[S(\sigma_{x\,B_1}) - S(\sigma_{x\,RB_1}) \right]}_{I_c(R\rangle B_1)_{\sigma_x}}$$

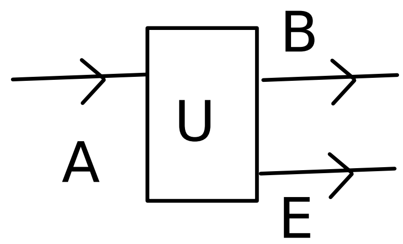
So, if Bob holds classical info, the coherent info is an average over his classical label.

Coherent info of a quantum channel:

For channel:

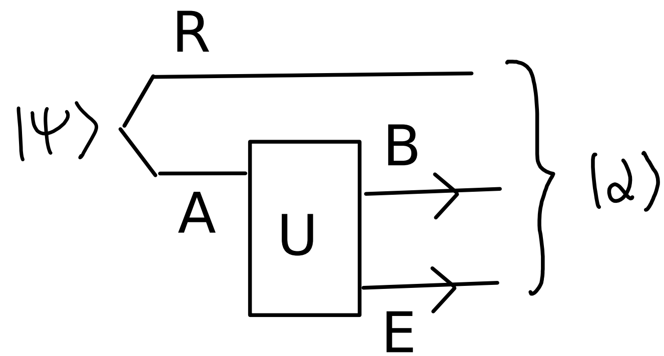


with isometric extension



for any input ρ_A purified to $|\psi\rangle_{RA}$

define $|\alpha\rangle_{RBE} = (I \otimes U)|\psi\rangle_{RA}$,



Def (1-shot coherent info of N):

$$Q^{(1)}(N) := \max_{|\psi\rangle} I_c(R\rangle B)_{I\otimes N(|\psi\rangle\langle\psi|)_{RB}}$$

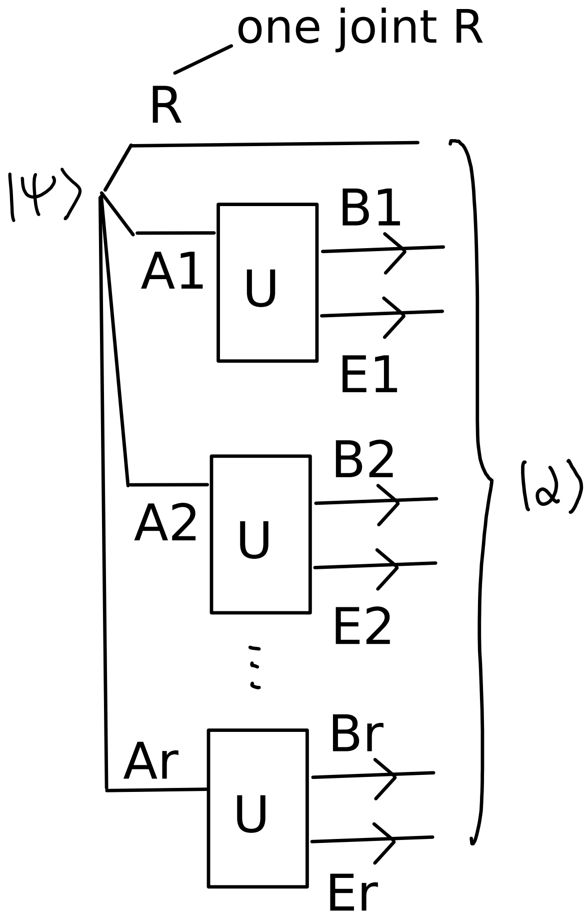
$$= \max_{|\psi\rangle} (S_B - S_E)_{|\alpha\rangle}$$

1/2 of max difference (over input) between S(R:B) & S(R:E)
made possible by 1 use of N

Def (r-shot coherent info of N):

$$Q^{(r)}(N) := \frac{1}{r} Q^{(1)}(N^{\otimes r}) = \frac{1}{r} I_c(R\rangle B_1 B_2 \cdots B_r)_{I\otimes N^{\otimes r}(|\psi\rangle\langle\psi|)}$$

$$|\psi\rangle_{RA_1\cdots A_r}$$



$$= \frac{1}{r}\left[S_{B_1B_2\cdots B_r} - S_{RB_1\cdots B_r}\right]_{|\alpha\rangle}$$

$$= \frac{1}{r}\left[S_{B_1B_2\cdots B_r} - S_{E_1\cdots E_r}\right]_{|\alpha\rangle}$$

The LSD thm: $Q(N) = \sup_r Q^{(r)}(N)$

Lloyd-96-Shor-02-Devetak-04