CO781 / QIC 890:

Theory of Quantum Communication

Topic 5, part 2

Transmitting quantum data through a quantum channels

- the proof of the LSD theorem outline
- the decoupling approach (exact)
- the decoupling approach (approx)
- the decoupling condition (1-shot)

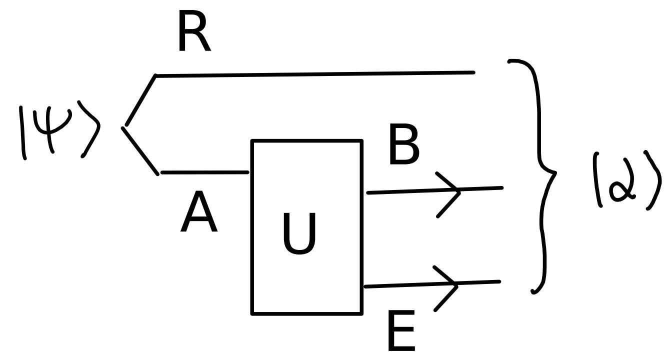
today

- the direct coding theorem for the LSD theorem
- typicality for the direct coding theorem
- the decoupling condition (applied to direct coding theorem)
- the converse

Last time:

For any input ρ_A purified to $|\psi\rangle_{RA}$

define $|\varphi\rangle_{RBE} = (I \otimes U)\,|\psi\rangle_{RA}$,



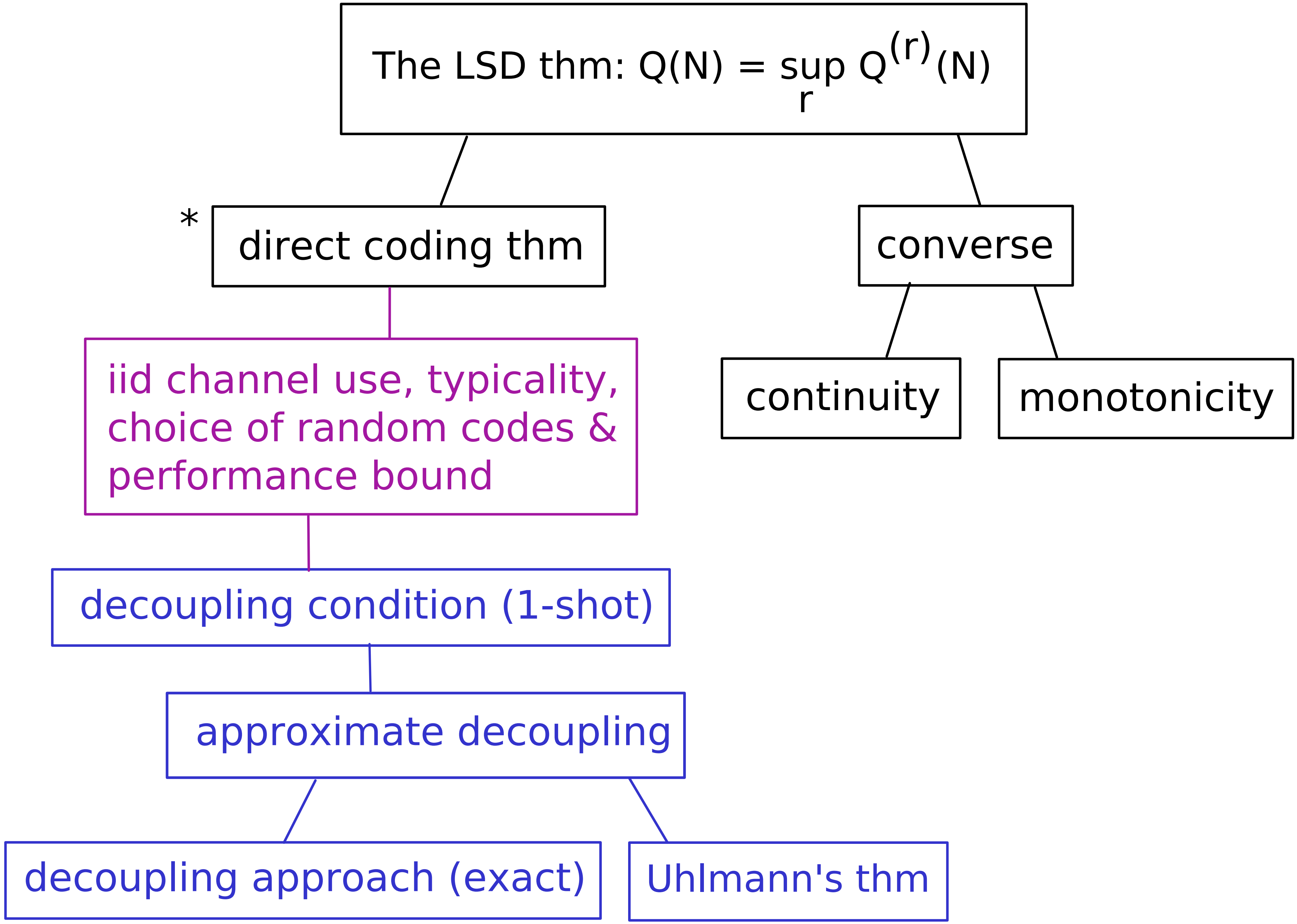
Def (1-shot coherent info of N):

$$Q^{(1)}(N) := \max_{|\psi\rangle} I_c(R\rangle B)_{I\otimes N(|\psi\rangle\langle\psi|)_{RB}}$$

$$= \max_{|\psi\rangle} (S_B - S_E)_{|\varphi\rangle}$$

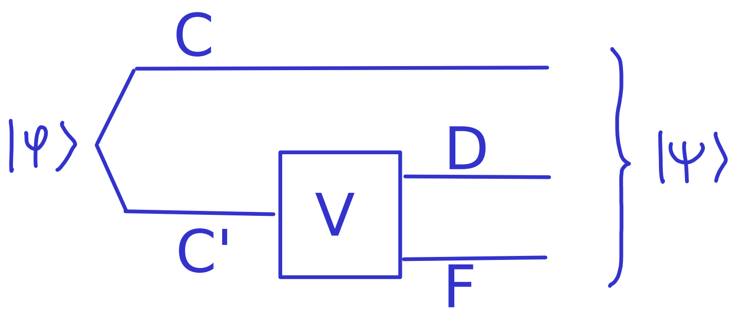
$$Q^{(r)}(N) := \frac{1}{r}\, Q^{(1)}(N^{\otimes r}) = \frac{1}{r} \underset{|\psi\rangle_{RA_1\cdots A_r}}{I_c}(R\rangle B_1 B_2 \cdots B_r)_{I\otimes N^{\otimes r}(|\psi\rangle\langle\psi|)}$$

The LSD thm: $Q(N) = \sup_r Q^{(r)}(N)$



* 0702005 Hayden, (M) Horodecki, Winter, Yard

Decoupling approach (exact case)



Shorthands: $\psi = |\psi\rangle\langle\psi|$

$\psi^{CF} = \mathrm{tr}_D\, |\psi\rangle\langle\psi|$

not to mean same ψ on C & F $\left\{ \begin{array}{l} \psi^{F} = \mathrm{tr}_{CD}\, |\psi\rangle\langle\psi| \\ \psi^{C} = \mathrm{tr}_{DF}\, |\psi\rangle\langle\psi| \end{array} \right.$

tripartite pure state on CDF
that arises from a pure state on CC' by splitting C' into DF

e.g., $|\varphi\rangle_{CC'} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

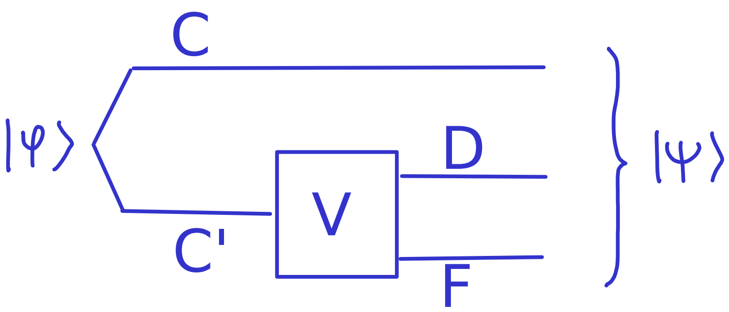
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle\overbrace{\left(\sqrt{\tfrac{2}{3}}\,|01\rangle + \sqrt{\tfrac{1}{3}}\,|12\rangle\right)}^{V|0\rangle} + |1\rangle\overbrace{\left(\sqrt{\tfrac{2}{3}}\,|21\rangle + \sqrt{\tfrac{1}{3}}\,|32\rangle\right)}^{V|1\rangle}\right]_{CDF}$$

Idea: $\psi^{CF} = \frac{1}{3}|0\rangle\langle 0| \otimes |1\rangle\langle 1| + \frac{1}{6}|0\rangle\langle 0| \otimes |2\rangle\langle 2| + \frac{1}{3}|1\rangle\langle 1| \otimes |1\rangle\langle 1| + \frac{1}{6}|1\rangle\langle 1| \otimes |2\rangle\langle 2|$

$$= \underbrace{\left(\tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1|\right)_C}_{\psi^C} \otimes \underbrace{\left(\tfrac{2}{3}|1\rangle\langle 1| + \tfrac{1}{3}|2\rangle\langle 2|\right)_F}_{\psi^F}$$

product state

Decoupling approach (exact case)



Shorthands: $\psi = |\psi\rangle\langle\psi|$

$$\psi^{CF} = \mathrm{tr}_D\, |\psi\rangle\langle\psi|$$

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$$\left\{ \begin{array}{l} \psi^{F} = \mathrm{tr}_{CD}\, |\psi\rangle\langle\psi| \\ \psi^{C} = \mathrm{tr}_{DF}\, |\psi\rangle\langle\psi| \end{array} \right.$$

tripartite pure state on CDF
that arises from a pure state on CC' by splitting C' into DF

e.g., $|\psi\rangle_{CC'} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

$V|0\rangle$ $V|1\rangle$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle\left(\sqrt{\tfrac{2}{3}}\,|01\rangle + \sqrt{\tfrac{1}{3}}\,|12\rangle\right) + |1\rangle\left(\sqrt{\tfrac{2}{3}}\,|21\rangle + \sqrt{\tfrac{1}{3}}\,|32\rangle\right)\right]_{CDF}$$

Idea: $\psi^{CF} = \left(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\right)_C \otimes \left(\frac{2}{3}|1\rangle\langle 1| + \frac{1}{3}|2\rangle\langle 2|\right)_F$

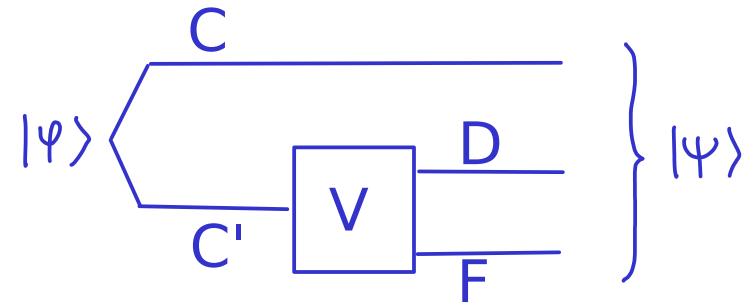
Map $|0\rangle_D \to |00\rangle_{D_1D_2}$, $|1\rangle_D \to |01\rangle_{D_1D_2}$, $|2\rangle_D \to |10\rangle_{D_1D_2}$, $|3\rangle_D \to |11\rangle_{D_1D_2}$,

$$|\psi\rangle \longrightarrow \frac{1}{\sqrt{2}}\left[|0\rangle\left(\sqrt{\tfrac{2}{3}}\,|001\rangle + \sqrt{\tfrac{1}{3}}\,|012\rangle\right) + |1\rangle\left(\sqrt{\tfrac{2}{3}}\,|101\rangle + \sqrt{\tfrac{1}{3}}\,|112\rangle\right)\right]_{CD_1D_2F}$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)_{CD_1} \otimes \left(\sqrt{\tfrac{2}{3}}\,|01\rangle + \sqrt{\tfrac{1}{3}}\,|12\rangle\right)_{D_2F}$$

* If tracing out D uncorrelates CF, can rotate D to uncorrelate the purification

Decoupling approach (exact case)



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$\psi^{CF} = \mathrm{tr}_D\, |\psi\rangle\langle\psi|$

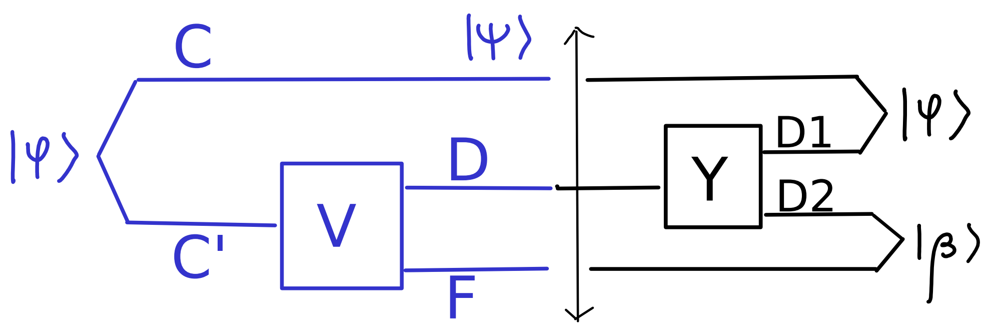
$\psi^{F} = \mathrm{tr}_{CD}\, |\psi\rangle\langle\psi|$

$\psi^{C} = \mathrm{tr}_{DF}\, |\psi\rangle\langle\psi|$

Lemma: if $\psi^{CF} = \psi^{C} \otimes \psi^{F}$

then $\exists$ isometry Y: D $\rightarrow$ D1 D2,

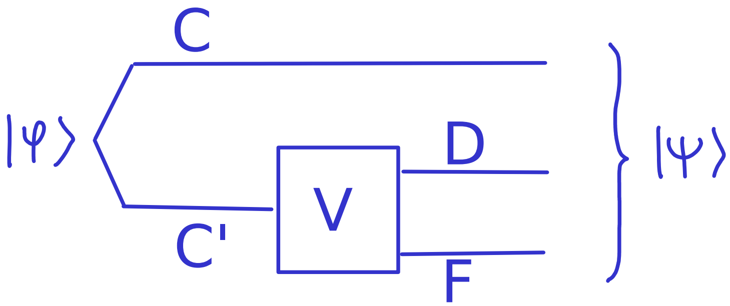
$$I_{CF} \otimes Y_D\, |\psi\rangle = |\varphi\rangle_{CD_1} \otimes |\beta\rangle_{D_2 F}$$ where $\beta^F = \psi^F$.



Interpretation:
C is reference to C'
channel V takes input C'
to output D & env F.

If env F is uncorrelated with reference C, then, the output D can be decoded to recover the state on CC' (now on CD1), reverting the noisy quantum channel.

Decoupling approach (exact case)



Shorthands: $\psi = |\psi\rangle\langle\psi|$

$\psi^{CF} = \mathrm{tr}_D\, |\psi\rangle\langle\psi|$

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Lemma: if $\psi^{CF} = \psi^C \otimes \psi^F$

then $\exists$ isometry $Y: D \rightarrow D1\, D2$,

$$I_{CF} \otimes Y_D |\psi\rangle = |\varphi\rangle_{CD_1} \otimes |\beta\rangle_{D_2F} \quad \text{where } \beta^F = \psi^F.$$

Proof: Consider 2 possible purifications of ψ^{CF}

$|\psi\rangle_{CDF}$ and $|\varphi\rangle_{CD_1} \otimes |\beta\rangle_{D_2F}$

any 2 purifications are related by an isometry between the purifying systems, so, $\exists\, Y: D \rightarrow D_1 D_2$ so that

$$I_{CF} \otimes Y_D |\psi\rangle = |\varphi\rangle_{CD_1} \otimes |\beta\rangle_{D_2F}$$

Decoupling approach (approximate case via Uhlmann's thm)

Recall Uhlmann's theorem:

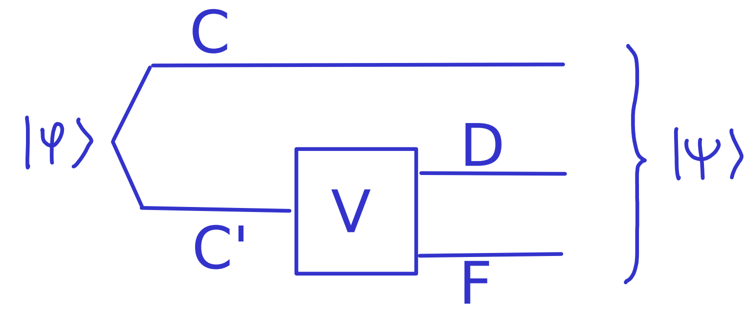
For two states ρ, σ on the same system, their fidelity

$$F(\rho, \sigma) = \max |\langle \mu | \nu \rangle|$$

where $|\nu\rangle$ is any fixed purification of σ

and the max is over all possible purifications $|\mu\rangle$ of ρ.

Decoupling approach (approximate case via Uhlmann's thm)



Lemma: if $\| \psi^{CF} - \alpha^{C} \otimes \beta^{F} \|_1 \le \epsilon$

then $\exists$ isometry $Y: D \to D1\ D2$,

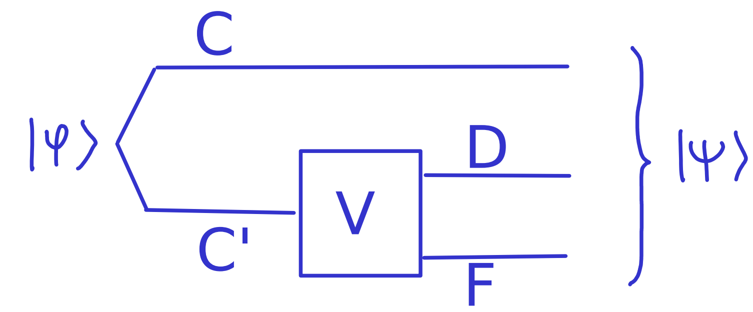
$F(I_{CF} \otimes Y_D |\psi\rangle, |\alpha\rangle_{CD_1} \otimes |\beta\rangle_{D_2 F}) \ge 1 - \frac{\epsilon}{2}$

easily converted to trace distance

Proof: $\| \psi^{CF} - \alpha^{C} \otimes \beta^{F} \|_1 \le \epsilon \Rightarrow F(\psi^{CF}, \alpha^{C} \otimes \beta^{F}) \ge 1 - \frac{\epsilon}{2}$

Nielsen & Chuang 1st Ed (9.109)

Decoupling approach (approximate case via Uhlmann's thm)



Lemma: if $\| \psi^{CF} - \alpha^{C} \otimes \beta^{F} \|_1 \leq \epsilon$

then $\exists$ isometry $Y: D \to D_1 D_2$,

$F(I_{CF} \otimes Y_D |\psi\rangle , |\alpha\rangle_{CD_1} \otimes |\beta\rangle_{D_2 F}) \geq 1 - \frac{\epsilon}{2}$

Proof: $\| \psi^{CF} - \alpha^{C} \otimes \beta^{F} \|_1 \leq \epsilon \Rightarrow F(\psi^{CF}, \alpha^{C} \otimes \beta^{F}) \geq 1 - \frac{\epsilon}{2}$

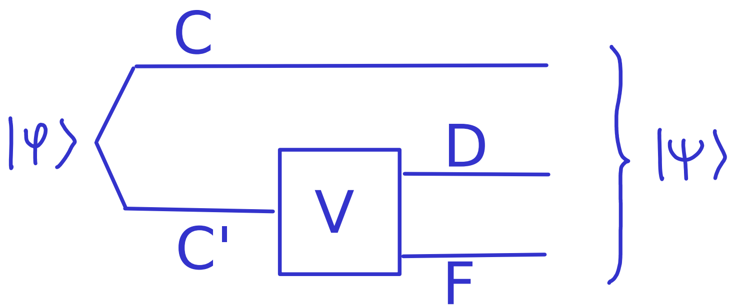
// Uhlmann's Thm

max over $|\mu\rangle$ that purifies ψ^{CF} $\quad \left| \langle \mu | \cdot |\alpha\rangle_{CD_1} \otimes |\beta\rangle_{D_2 F} \right| \leftarrow \star$

Let $|\mu^*\rangle$ attain the max. Any 2 purifications of ψ^{CF} are related by an isometry on the purifying system. $\because$ $|\psi\rangle, |\mu^*\rangle$ both purify ψ^{CF},

$\exists Y: D \to D_1 D_2$, $|\mu^*\rangle = I_{CF} \otimes Y_D |\psi\rangle$. Sub into $\star$ proves the lemma.

Decoupling approach (approximate case via Uhlmann's thm)



Lemma: if $\| \psi^{CF} - \alpha^C \otimes \beta^F \|_1 \leqslant \epsilon$

then $\exists$ isometry Y: D $\rightarrow$ D1 D2,

$F(I_{CF} \otimes Y_D |\psi\rangle , |\alpha\rangle_{CD_1} \otimes |\beta\rangle_{D_2F}) \geqslant 1 - \frac{\epsilon}{2}$

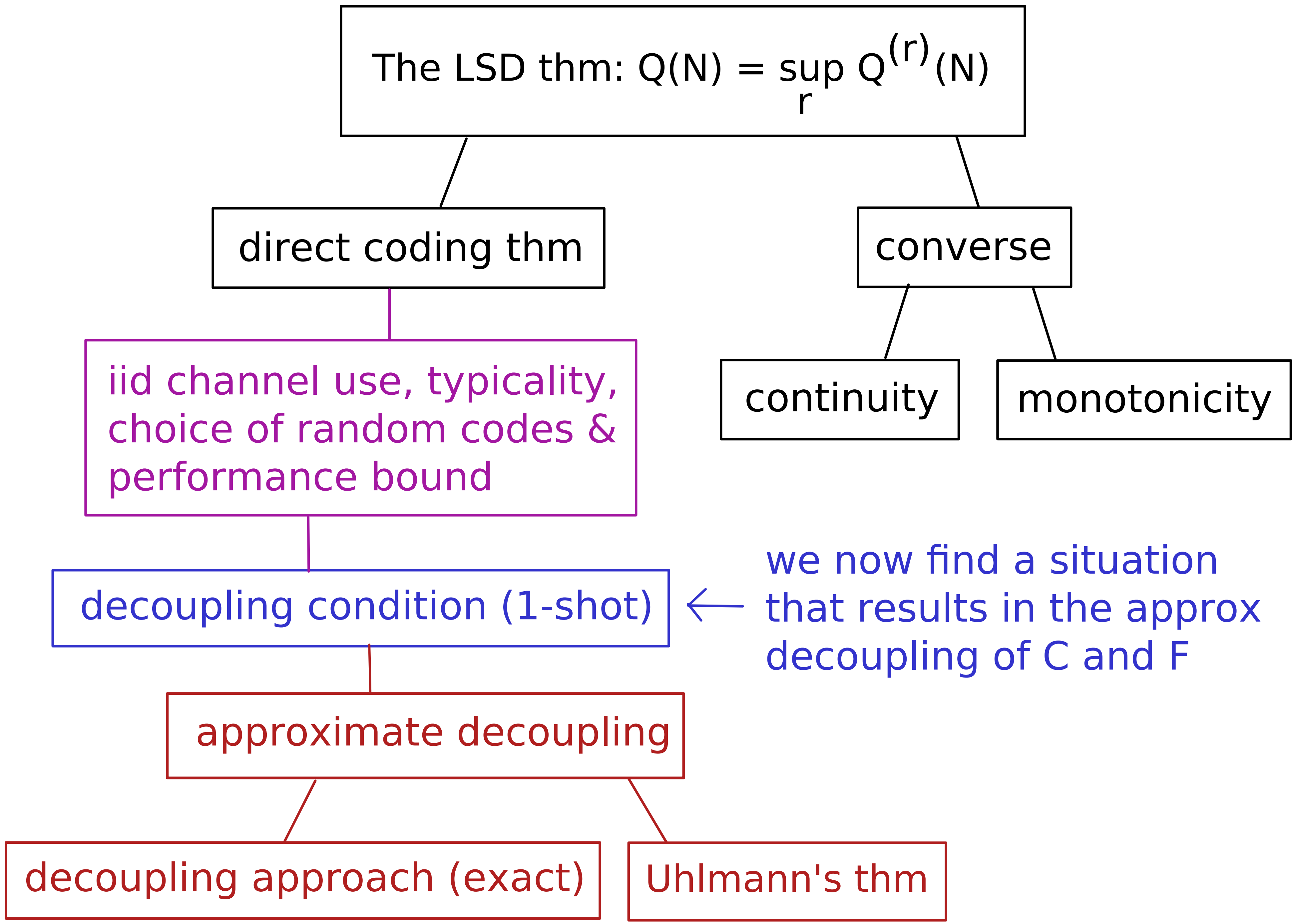
Note that with the approx, we do not demand $\alpha^C = \psi^C$

this formulation is more convenient later.

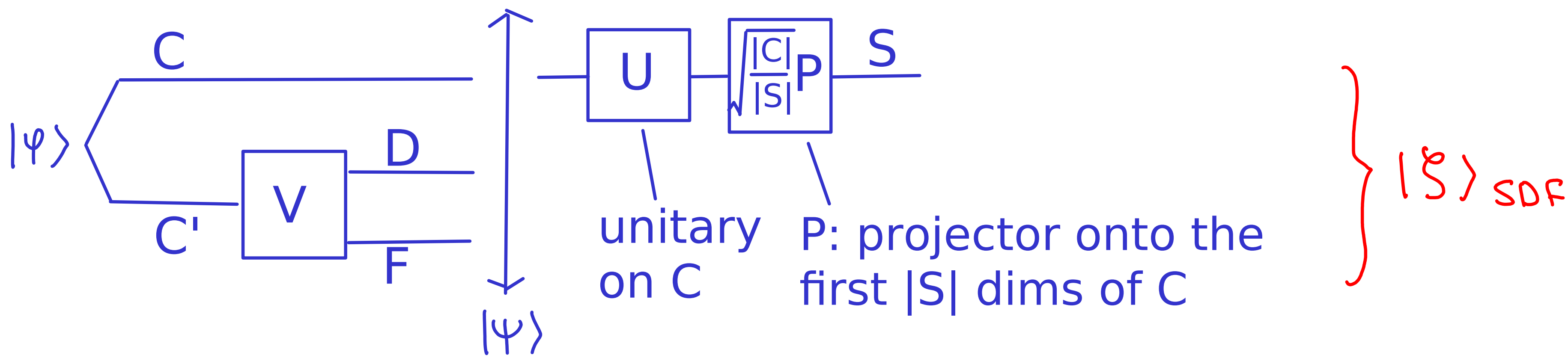
Also, the above is very similar to setting $\alpha^C = \psi^C$

since, by monotonicity of trace distance under partial tracing:

$$\| \psi^{CF} - \alpha^C \otimes \beta^F \|_1 \leqslant \epsilon \Rightarrow \| \psi^C - \alpha^C \|_1 \leqslant \epsilon$$


The LSD thm: $Q(N) = \sup_r Q^{(r)}(N)$
direct coding thm
converse
iid channel use, typicality, choice of random codes & performance bound
continuity
monotonicity
decoupling condition (1-shot)
we now find a situation that results in the approx decoupling of C and F
approximate decoupling
decoupling approach (exact)
Uhlmann's thm

Decoupling condition (1-shot):



Let $|\xi\rangle = \sqrt{\frac{|C|}{|S|}}\, P U |\psi\rangle$ (a vector not normalized, on SDF)

Theorem: if U chosen according to the Haar measure,

then $$\mathbb{E}_U \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\mathrm{Tr}\left[(\psi^D)^2 \right] \right)^{\frac{1}{2}}$$

not normalized

max mixed state on S

Want $\mathbb{E}_U \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\mathrm{Tr}\left[(\psi^D)^2 \right] \right)^{\frac{1}{2}}$

not normalized   max mixed state on S

Proof:

(1) replace 1-norm by 2-norm, using Cauchy-Schwartz ineq

$$\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2$$

Elaborate (1):

CS ineq: $\sum_i |x_i| |y_i| \leq \sqrt{\sum_i x_i^2} \sqrt{\sum_j y_j^2}$

For any hermitian dxd matrix M, let $\{x_i\}$ = eigenvalues of M

let $y_j = 1$ for all j

Then $\|M\|_1 = \sum_i |x_i| = \sum_i |x_i| |y_i| \underset{CS}{\leq} \sqrt{\sum_i x_i^2} \sqrt{\sum_j y_j^2} = \|M\|_2 \sqrt{d}$

Want $\mathbb{E}_U \left\| \varrho^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\mathrm{Tr}\left[(\psi^D)^2 \right] \right)^{\frac{1}{2}}$

not normalized — max mixed state on S

Proof:

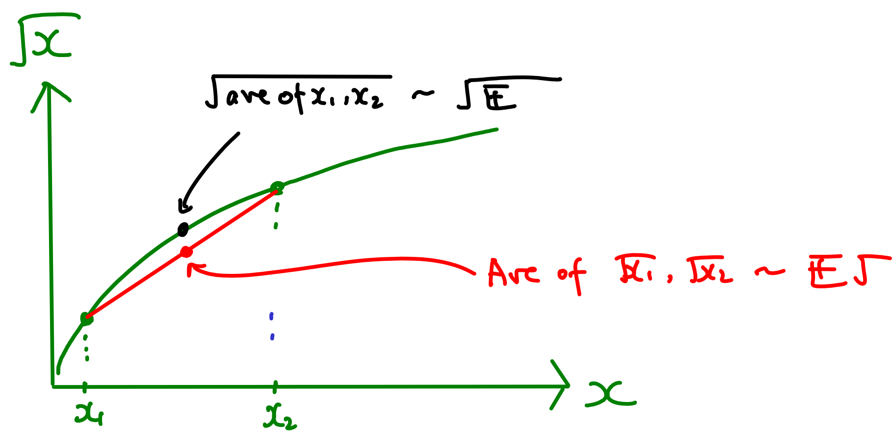
(1) replace 1-norm by 2-norm, using Cauchy-Schwartz ineq

$$\left\| \varrho^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left\| \varrho^{SF} - \pi^S \otimes \psi^F \right\|_2$$

(2) use concavity of square root

$$\mathbb{E}_U \sqrt{\left\| \varrho^{SF} - \pi^S \otimes \psi^F \right\|_2^2} \leq \sqrt{\mathbb{E}_U \left\| \varrho^{SF} - \pi^S \otimes \psi^F \right\|_2^2}$$

Elaborate (2):



Want $\mathbb{E}_U \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\mathrm{Tr}\left[(\psi^D)^2 \right] \right)^{\frac{1}{2}}$

not normalized

max mixed state on S

Proof:

(1) replace 1-norm by 2-norm, using Cauchy-Schwartz ineq

$$\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2$$

(2) use concavity of square root

$$\mathbb{E}_U \sqrt{\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2} \leq \sqrt{\mathbb{E}_U \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2}$$

(3) expand

$$\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2 = \mathrm{tr} \left(\xi^{SF} - \pi^S \otimes \psi^F \right)\left(\xi^{SF} - \pi^S \otimes \psi^F \right)$$

Elaborate (3):

$$\|\xi^{SF} - \pi^S\otimes\psi^F\|_2^2 = \mathrm{tr}\left(\xi^{SF} - \pi^S\otimes\psi^F\right)\left(\xi^{SF} - \pi^S\otimes\psi^F\right)$$

$$= \mathrm{tr}\left(\xi^{SF}\right)^2 - 2\underbrace{\mathrm{tr}\left(\pi^S\otimes\psi^F\right)\xi^{SF}}_{\frac{1}{|S|}\mathrm{tr}\,\psi^F\,\xi^F} + \underbrace{\mathrm{tr}\left(\pi^S\otimes\psi^F\right)^2}_{\mathrm{tr}\left[\frac{I}{|S|^2}\otimes(\psi^F)^2\right]}$$

on F

$\cancel{\frac{1}{|S|}\mathrm{tr}(\psi^F)^2}$

$$\mathrm{tr}\left[\frac{I}{|S|^2}\otimes(\psi^F)^2\right] = \frac{1}{|S|}\mathrm{tr}(\psi^F)^2$$

Useful fact on partial trace:

$$\mathrm{tr}\,(I_1\otimes K_2)\,M_{12} = \mathrm{tr}\,K\,\underbrace{(\mathrm{tr}_1 M_{12})}$$

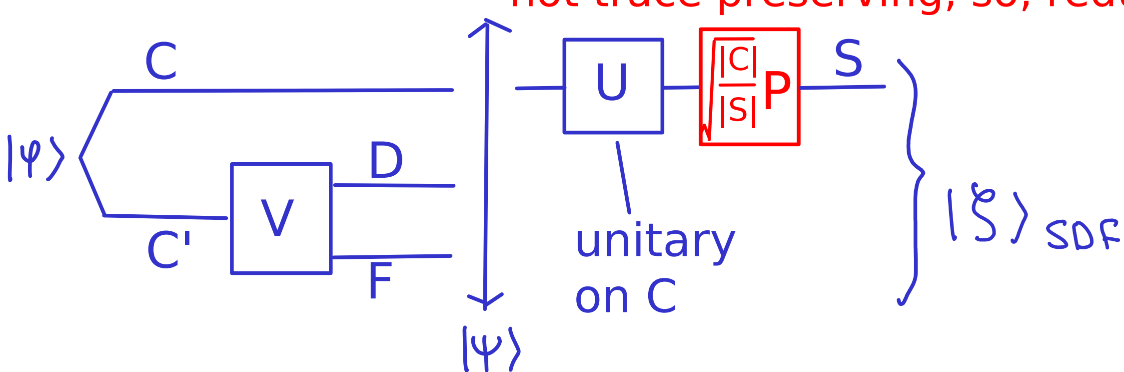
on sys 1,2

on sys 2

NB $\pi^S = \frac{I}{|S|}$

$\therefore$

not trace preserving, so, reduced state on F need not be preserved



NB $\cancel{\psi^F = \xi^F}$

$\therefore$

Solution: we later average the above expression over the choice of U.

$$\mathbb{E}_U \frac{1}{|S|}\mathrm{tr}\,\psi^F\xi^F = \frac{1}{|S|}\mathrm{tr}\,\psi^F\,\mathbb{E}_U\,\xi^F$$

next page: $\mathbb{E}_U\,\xi^F = \psi^F$

so same conclusion with E-over-U.

$$\xi^{SF} = \frac{|C|}{|S|}\left(PU \otimes I_F\right)\psi^{CF}\left(U^{\dagger}P \otimes I_F\right)$$

$$\mathop{\mathbb{E}}_{U}\xi^{F} = \mathop{\mathbb{E}}_{U}\operatorname{tr}_S \frac{|C|}{|S|}\left(PU \otimes I_F\right)\psi^{CF}\left(U^{\dagger}P \otimes I_F\right)$$

note P is indep of U so we can take the E-U inside

$$= \operatorname{tr}_S \frac{|C|}{|S|}\left(P \otimes I_F\right)\underbrace{\mathop{\mathbb{E}}_{U}\left(U \otimes I_F\right)\psi^{CF}\left(U^{\dagger} \otimes I_F\right)}_{\frac{I}{|C|}\otimes\psi^{F}}\left(P \otimes I_F\right)$$

$$= \operatorname{tr}_S \frac{\cancel{|C|}}{|S|}\left(P \otimes I_F\right)\frac{I}{\cancel{|C|}}\otimes\psi^{F}\left(P \otimes I_F\right) = \psi^{F}$$

so everything is fine !

Elaborate (3):

$$\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2 = \mathrm{tr}\left(\xi^{SF} - \pi^S \otimes \psi^F\right)\left(\xi^{SF} - \pi^S \otimes \psi^F\right)$$

$$= \mathrm{tr}\left(\xi^{SF}\right)^2 - 2\,\underbrace{\mathrm{tr}\left(\pi^S \otimes \psi^F\right)\xi^{SF}}_{\frac{1}{|S|}\underbrace{\mathrm{tr}\,\psi^F\,\xi^F}_{\text{on } F}} + \underbrace{\mathrm{tr}\left(\pi^S \otimes \psi^F\right)^2}_{\mathrm{tr}\left[\frac{I}{|S|^2}\otimes\left(\psi^F\right)^2\right] = \frac{1}{|S|}\mathrm{tr}\left(\psi^F\right)^2}$$

$$\mathbb{E}_u \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2$$

$$= \mathbb{E}_u\, \mathrm{tr}\left(\xi^{SF}\right)^2 - 2\,\frac{1}{|S|}\,\mathrm{tr}\,\psi^F\, \mathbb{E}_u\, \xi^F + \frac{1}{|S|}\,\mathrm{tr}\left(\psi^F\right)^2$$

$$= \mathbb{E}_u\, \mathrm{tr}\left(\xi^{SF}\right)^2 - \frac{1}{|S|}\,\mathrm{tr}\left(\psi^F\right)^2$$

Want $\mathbb{E}_U \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\mathrm{Tr}\left[(\psi^D)^2 \right] \right)^{\frac{1}{2}}$

not normalized

max mixed state on S

Proof:

(1) replace 1-norm by 2-norm, using Cauchy-Schwartz ineq

$$\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2$$

(2) use concavity of square root

$$\mathbb{E}_U \sqrt{\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2} \leq \sqrt{\mathbb{E}_U \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2}$$

(3) expand $\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2 = \mathrm{tr} \left(\xi^{SF} \right)^2 - \frac{1}{|S|} \mathrm{tr} (\psi^F)^2$

get $\mathbb{E}_U \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2 = \mathbb{E}_U \mathrm{tr} \left(\xi^{SF} \right)^2 - \frac{1}{|S|} \mathrm{tr} (\psi^F)^2$

(4) evaluate $\mathbb{E}_U \mathrm{tr} \left(\xi^{SF} \right)^2$

Elaborate (4):

$$\mathbb{E}_U \, \mathrm{tr} \left(\xi^{SF}\right)^2 = \mathbb{E}_U \, \mathrm{tr}_{S_1F_1S_2F_2} \left(\xi^{S_1F_1} \otimes \xi^{S_2F_2}\right) \Sigma^{S_1F_1\,S_2F_2}$$

swapping S1 F1 with S2 F2

$$\Sigma^{S_1F_1\,S_2F_2} = \Sigma^{S_1S_2} \otimes \Sigma^{F_1F_2}$$

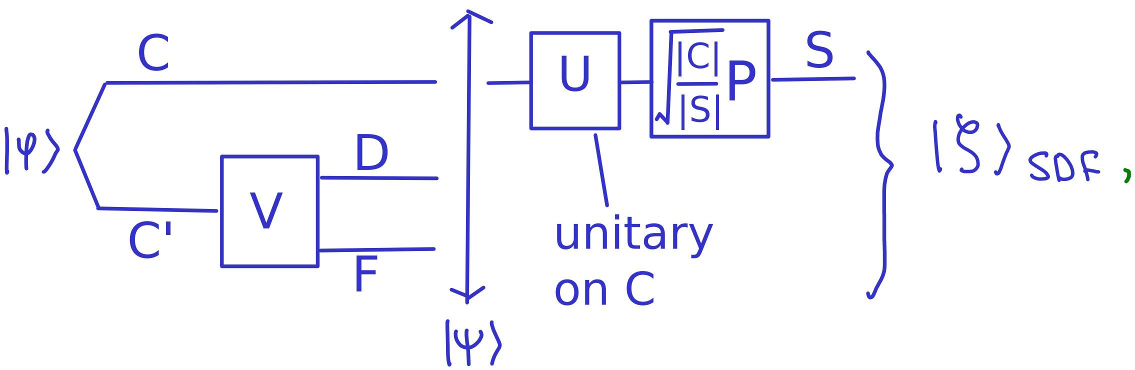
A neat trick:

$$\begin{aligned}
\mathrm{tr}\left(M \otimes N\right) \overbrace{\left(\sum_{ij} |ij\rangle\langle ji|\right)}^{\Sigma} &= \sum_{ij} \langle ji|\, M \otimes N \,|ij\rangle \\
&= \sum_{ij} \langle j|M|i\rangle \langle i|N|j\rangle \\
&= \sum_{j} \sum_{i} \langle j|M|i\rangle \langle i|N|j\rangle \\
&= \mathrm{tr}\, MN
\end{aligned}$$

Elaborate (4):

$$\mathbb{E}_U \operatorname{tr}\left(\xi^{SF}\right)^2 = \mathbb{E}_U \operatorname{tr}_{S_1F_1S_2F_2}\left(\xi^{S_1F_1}\otimes\xi^{S_2F_2}\right)\left(\Sigma^{S_1S_2}\otimes\Sigma^{F_1F_2}\right)$$

$$= \mathbb{E}_U \operatorname{tr}_{S_1F_1S_2F_2}\frac{|C|^2}{|S|^2}\left(PU\otimes PU\otimes I_{F_1F_2}\right)\psi^{C_1F_1}\otimes\psi^{C_2F_2}\left(U^\dagger P\otimes U^\dagger P\otimes I_{F_1F_2}\right)\left(\Sigma^{S_1S_2}\otimes\Sigma^{F_1F_2}\right)$$



$$\therefore\ \xi^{SF} = \frac{|C|}{|S|}\left(PU\otimes I_F\right)\psi^{CF}\left(U^\dagger P\otimes I_F\right)$$

$$\xi^{S_1F_1}\otimes\xi^{S_2F_2} = \frac{|C|^2}{|S|^2}\left(PU\otimes PU\otimes I_{F_1F_2}\right)\psi^{C_1F_1}\otimes\psi^{C_2F_2}\left(U^\dagger P\otimes U^\dagger P\otimes I_{F_1F_2}\right)$$

Elaborate (4):

$$\mathbb{E}_U \, \mathrm{tr}\left(\xi^{SF}\right)^2 = \mathbb{E}_U \, \mathrm{tr}_{S_1F_1S_2F_2}\left(\xi^{S_1F_1}\otimes\xi^{S_2F_2}\right)\left(\Sigma^{S_1S_2}\otimes\Sigma^{F_1F_2}\right)$$

$$= \mathbb{E}_U \, \mathrm{tr}_{S_1F_1S_2F_2}\frac{|C|^2}{|S|^2}\left(PU\otimes PU\otimes I_{F_1F_2}\right)\psi^{C_1F_1}\otimes\psi^{C_2F_2}\left(U^\dagger P\otimes U^\dagger P\otimes I_{F_1F_2}\right)\left(\Sigma^{S_1S_2}\otimes\Sigma^{F_1F_2}\right)$$

cyclic prop of trace

$$= \mathbb{E}_U \, \mathrm{tr}_{S_1F_1S_2F_2}\frac{|C|^2}{|S|^2}\psi^{C_1F_1}\otimes\psi^{C_2F_2}\left(U^\dagger P\otimes U^\dagger P\otimes I_{F_1F_2}\right)\left(\Sigma^{S_1S_2}\otimes\Sigma^{F_1F_2}\right)\left(PU\otimes PU\otimes I_{F_1F_2}\right)$$

E & tr commute

Ψ indep of U

multiply and simplify

$$= \mathrm{tr}_{C_1F_1C_2F_2}\frac{|C|^2}{|S|^2}\psi^{C_1F_1}\otimes\psi^{C_2F_2}\;\underbrace{\mathbb{E}_U\left[\left(U^\dagger P\otimes U^\dagger P\right)\Sigma^{S_1S_2}\left(PU\otimes PU\right)\right]}_{\left(\frac{|S|}{|C|}\right)^2\left(\frac{1-\frac{1}{|C||S|}}{1-\frac{1}{|C|^2}}\right)\Sigma^{C_1C_2}+\frac{|S|}{|C|^2}\left(\frac{1-\frac{|S|}{|C|}}{1-\frac{1}{|C|^2}}\right)I^{C_1C_2}}\otimes\Sigma^{F_1F_2}$$

(see 0702005,
or 1501.04592,
or Watrous bk)

$$= \mathrm{tr}_{C_1F_1C_2F_2}\,\psi^{C_1F_1}\otimes\psi^{C_2F_2}\left[\left(\frac{1-\frac{1}{|C||S|}}{1-\frac{1}{|C|^2}}\right)\Sigma^{C_1C_2}+\frac{1}{|S|}\left(\frac{1-\frac{|S|}{|C|}}{1-\frac{1}{|C|^2}}\right)I^{C_1C_2}\right]\otimes\Sigma^{F_1F_2}$$

$$= \mathrm{tr}_{C_1F_1C_2F_2}\, \psi^{C_1F_1} \otimes \psi^{C_2F_2} \left[\left(\frac{1 - \frac{1}{|C||S|}}{1 - \frac{1}{|C|^2}} \right) \Sigma^{C_1C_2} + \frac{1}{|S|} \left(\frac{1 - \frac{|S|}{|C|}}{1 - \frac{1}{|C|^2}} \right) I^{C_1C_2} \right] \otimes \Sigma^{F_1F_2}$$

$$= \left(\frac{1 - \frac{1}{|C||S|}}{1 - \frac{1}{|C|^2}} \right) \mathrm{tr}_{C_1F_1C_2F_2}\, \psi^{C_1F_1} \otimes \psi^{C_2F_2}\, \Sigma^{C_1C_2} \otimes \Sigma^{F_1F_2}$$

neat trick

$$+ \frac{1}{|S|} \left(\frac{1 - \frac{|S|}{|C|}}{1 - \frac{1}{|C|^2}} \right) \mathrm{tr}_{C_1F_1C_2F_2}\, \psi^{C_1F_1} \otimes \psi^{C_2F_2}\, I^{C_1C_2} \otimes \Sigma^{F_1F_2}$$

partial trace

$$= \underbrace{\left(\frac{1 - \frac{1}{|C||S|}}{1 - \frac{1}{|C|^2}} \right)}_{\leq 1} \mathrm{tr}_{CF} \left(\psi^{CF}\right)^2 + \frac{1}{|S|} \underbrace{\left(\frac{1 - \frac{|S|}{|C|}}{1 - \frac{1}{|C|^2}} \right)}_{\leq 1} \underbrace{\mathrm{tr}_{F_1F_2} \left(\underbrace{\mathrm{tr}_{C_1C_2}\, \psi^{C_1F_1} \otimes \psi^{C_2F_2}}_{\psi^{F_1} \otimes \psi^{F_2}} \right) \Sigma^{F_1F_2}}_{\mathrm{tr}\,(\psi^F)^2}$$

neat trick

$$\leq \mathrm{tr}_{CF} \left(\psi^{CF}\right)^2 + \frac{1}{|S|} \mathrm{tr}\,(\psi^F)^2$$

Want $\mathbb{E}_{U} \| \xi^{SF} - \pi^{S} \otimes \psi^{F} \|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\mathrm{Tr}\left[(\psi^{D})^2 \right] \right)^{\frac{1}{2}}$

got:

(1) replace 1-norm by 2-norm, using Cauchy-Schwartz ineq

$$\| \xi^{SF} - \pi^{S} \otimes \psi^{F} \|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \| \xi^{SF} - \pi^{S} \otimes \psi^{F} \|_2$$

(2) use concavity of square root

$$\mathbb{E}_{U} \sqrt{\| \xi^{SF} - \pi^{S} \otimes \psi^{F} \|_2^2} \leq \sqrt{\mathbb{E}_{U} \| \xi^{SF} - \pi^{S} \otimes \psi^{F} \|_2^2}$$

(3) $\mathbb{E}_{U} \| \xi^{SF} - \pi^{S} \otimes \psi^{F} \|_2^2 = \mathbb{E}_{U} \mathrm{tr} \left(\xi^{SF} \right)^2 - \frac{1}{|S|} \mathrm{tr} (\psi F)^2$

(4) evaluate $\mathbb{E}_{U} \mathrm{tr} \left(\xi^{SF} \right)^2 \leq \mathrm{tr}_{CF} \left(\psi^{CF} \right)^2 + \frac{1}{|S|} \mathrm{tr} (\psi F)^2$

together:

$$\mathbb{E}_{U} \| \xi^{SF} - \pi^{S} \otimes \psi^{F} \|_1 \overset{(1)}{\leq} \mathbb{E}_{U} |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \| \xi^{SF} - \pi^{S} \otimes \psi^{F} \|_2$$

$$\overset{(2)}{\leq} |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \sqrt{\mathbb{E}_{U} \| \xi^{SF} - \pi^{S} \otimes \psi^{F} \|_2^2}$$

(3) expand $\left\| \xi^{SF} - \pi^{S} \otimes \psi^{F} \right\|_2^2 = \mathrm{tr}\left(\xi^{SF}\right)^2 - \frac{1}{|S|}\mathrm{tr}\left(\psi^{F}\right)^2$

(4) evaluate $\mathop{\mathbb{E}}_{U} \mathrm{tr}\left(\xi^{SF}\right)^2 \leq \mathrm{tr}_{CF}\left(\psi^{CF}\right)^2 + \frac{1}{|S|}\mathrm{tr}\left(\psi^{F}\right)^2$

from (3)&(4):

indep of U

$$\mathop{\mathbb{E}}_{U} \left\| \xi^{SF} - \pi^{S} \otimes \psi^{F} \right\|_2^2 = \mathop{\mathbb{E}}_{U} \mathrm{tr}\left(\xi^{SF}\right)^2 - \frac{1}{|S|}\mathrm{tr}\left(\psi^{F}\right)^2$$

$$\leq \mathrm{tr}_{CF}\left(\psi^{CF}\right)^2 + \frac{1}{|S|}\mathrm{tr}\left(\psi^{F}\right)^2 - \frac{1}{|S|}\mathrm{tr}\left(\psi^{F}\right)^2$$

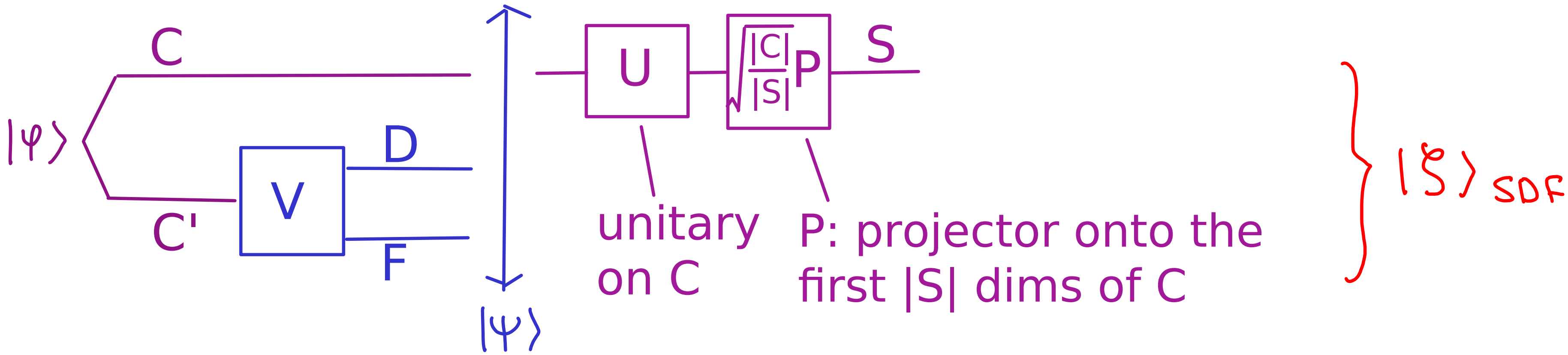
$$= \mathrm{tr}_{CF}\left(\psi^{CF}\right)^2 = \left(\mathrm{Tr}\left[\left(\psi^{D}\right)^2\right]\right)^{\frac{1}{2}}$$

returning to previous page:

$$\mathop{\mathbb{E}}_{U} \left\| \xi^{SF} - \pi^{S} \otimes \psi^{F} \right\|_1 \overset{(1)}{\leq} \mathop{\mathbb{E}}_{U} |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left\| \xi^{SF} - \pi^{S} \otimes \psi^{F} \right\|_2$$

$$\overset{(2)}{\leq} |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \sqrt{\mathop{\mathbb{E}}_{U} \left\| \xi^{SF} - \pi^{S} \otimes \psi^{F} \right\|_2^2} \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\mathrm{Tr}\left[\left(\psi^{D}\right)^2\right]\right)^{\frac{1}{2}} \qquad \square$$

Decoupling condition (1-shot):



Let $|\xi\rangle = \frac{|C|}{|S|} P\, U\, |\psi\rangle$ (a vector not normalized, on SDF)

Theorem: if U chosen according to the Haar measure,

then $\mathbb{E}_U \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\mathrm{Tr}\left[(\psi^D)^2 \right] \right)^{\frac{1}{2}}$

not normalized   max mixed state on S

Purple part generates a random code for channel V, code dim |S|. If the RHS of Theorem is small, using decoupling (approx), there exists U (or a code) s.t. V can be inverted by some Y on D.

Thur: make RHS of theorem small in (M,n) code for LSD thm.