

CO781 / QIC 890:

Theory of Quantum Communication

Topic 5, part 2

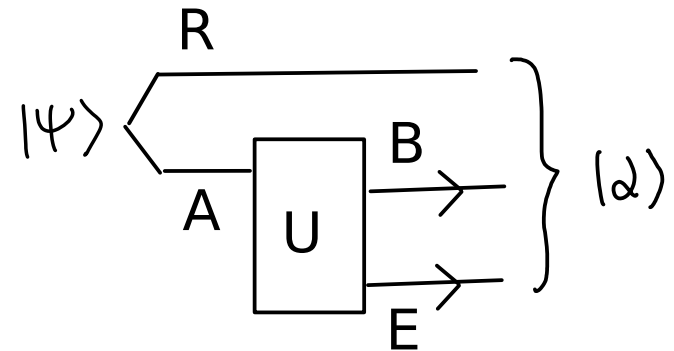
Transmitting quantum data through a quantum channels

- the proof of the LSD theorem outline
 - the decoupling approach (exact)
 - the decoupling approach (approx)
 - the decoupling condition (1-shot)
 - the direct coding theorem for the LSD theorem
 - typicality for the direct coding theorem
 - the decoupling condition (applied to direct coding theorem)
 - the converse
- } today

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Last time:

For any input ρ_A purified to $|\psi\rangle_{RA}$
 define $|\alpha\rangle_{RBE} = (\mathbb{I} \otimes U) |\psi\rangle_{RA}$,



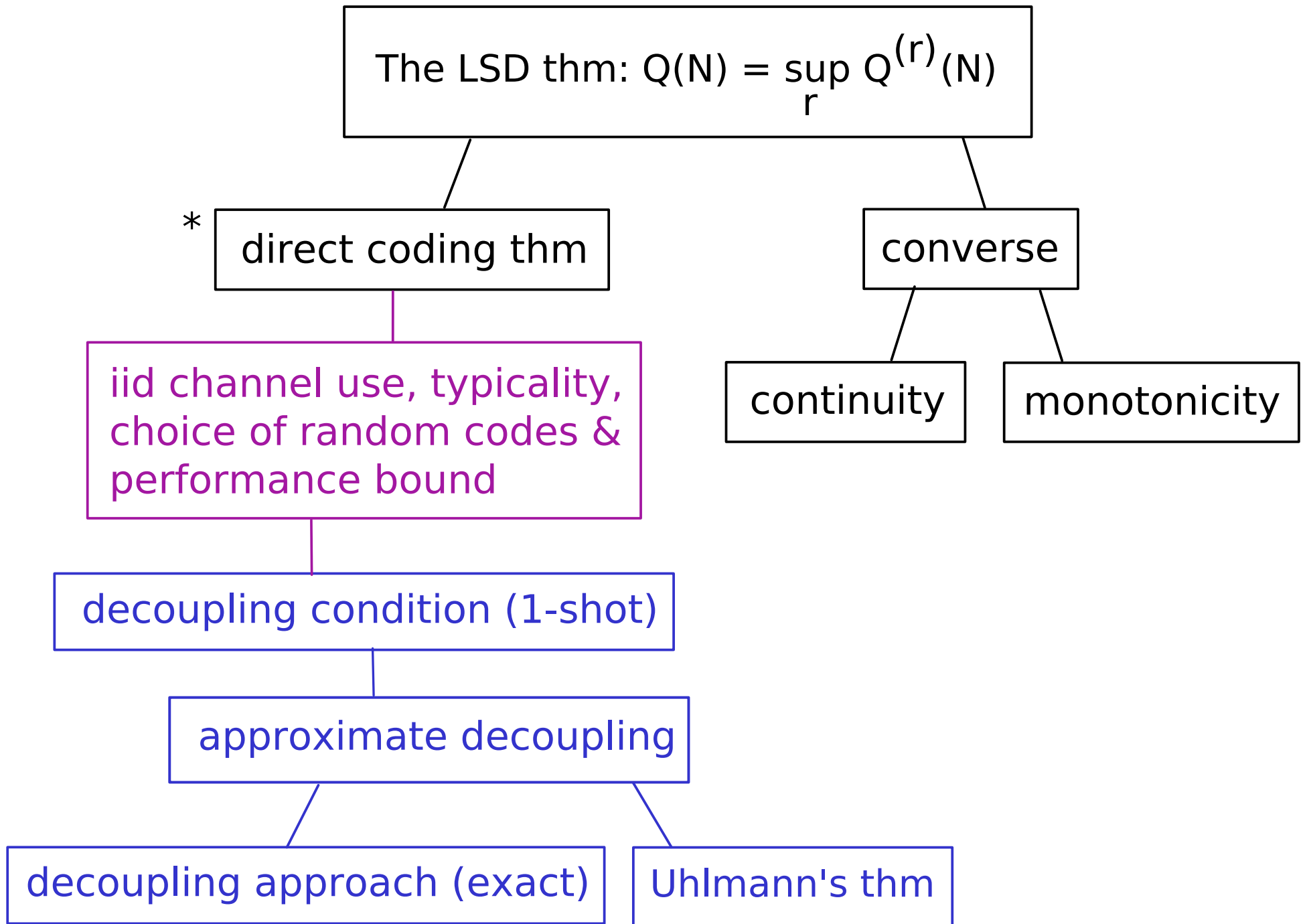
Def (1-shot coherent info of N):

$$Q^{(1)}(N) := \max_{|\psi\rangle} I_c(R \rangle B)_{\mathbb{I} \otimes N(|\psi\rangle\langle\psi|)_{RB}}$$

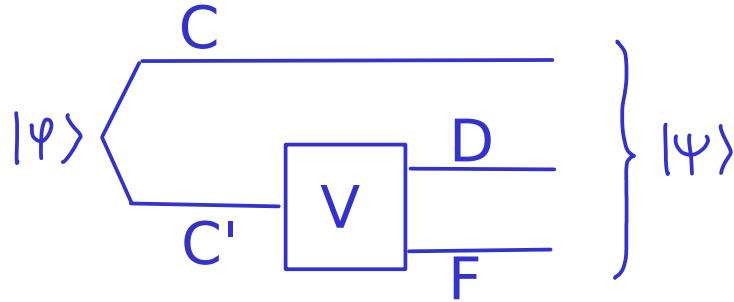
$$= \max_{|\psi\rangle} (S_B - S_E)_{|\alpha\rangle}$$

$$Q^{(r)}(N) := \frac{1}{r} Q^{(1)}(N^{\otimes r}) = \frac{1}{r} I_c(R \rangle B_1 B_2 \dots B_r)_{\mathbb{I} \otimes N^{\otimes r}(|\psi\rangle\langle\psi|)_{RA_1 \dots A_r}}$$

The LSD thm: $Q(N) = \sup_r Q^{(r)}(N)$



Decoupling approach (exact case)



Shorthands: $\Psi = |\psi\rangle\langle\psi|$

not to mean same Ψ on C & F

$$\Psi^{CF} = \text{tr}_D |\psi\rangle\langle\psi|$$

$$\Psi^F = \text{tr}_{CD} |\psi\rangle\langle\psi|$$

$$\Psi^C = \text{tr}_{DF} |\psi\rangle\langle\psi|$$

tripartite pure state on CDF

that arises from a pure state on CC' by splitting C' into DF

e.g., $|\psi\rangle_{CC'} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

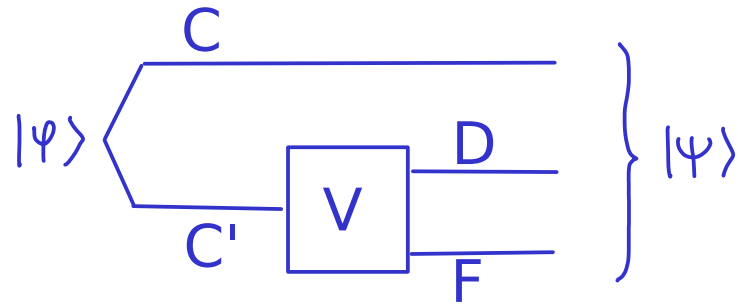
$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle \left(\sqrt{\frac{2}{3}} |01\rangle + \sqrt{\frac{1}{3}} |12\rangle \right) + |1\rangle \left(\sqrt{\frac{2}{3}} |21\rangle + \sqrt{\frac{1}{3}} |32\rangle \right) \right]_{CDF}$$

$\swarrow V|0\rangle$ $\swarrow V|1\rangle$
● ● ● ●

Idea: $\Psi^{CF} = \frac{1}{3} |0\rangle\langle 0| \otimes |1\rangle\langle 1| + \frac{1}{6} |0\rangle\langle 0| \otimes |2\rangle\langle 2| + \frac{1}{3} |1\rangle\langle 1| \otimes |1\rangle\langle 1| + \frac{1}{6} |1\rangle\langle 1| \otimes |2\rangle\langle 2|$

$$= \underbrace{\left(\frac{1}{2} |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1| \right)}_{\Psi^C} \otimes \underbrace{\left(\frac{2}{3} |1\rangle\langle 1| + \frac{1}{3} |2\rangle\langle 2| \right)}_{\Psi^F} \quad \text{product state}$$

Decoupling approach (exact case)



Shorthands: $\Psi = |\Psi\rangle\langle\Psi|$

$$\begin{aligned} \Psi^{CF} &= \text{tr}_D |\Psi\rangle\langle\Psi| \\ \text{not to mean same } \Psi \text{ on C \& F} &\left\{ \begin{aligned} \Psi^F &= \text{tr}_{CD} |\Psi\rangle\langle\Psi| \\ \Psi^C &= \text{tr}_{DF} |\Psi\rangle\langle\Psi| \end{aligned} \right. \end{aligned}$$

tripartite pure state on CDF

that arises from a pure state on CC' by splitting C' into DF

e.g., $|\Psi\rangle_{CC'} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle \left(\sqrt{\frac{2}{3}} |01\rangle + \sqrt{\frac{1}{3}} |12\rangle \right) + |1\rangle \left(\sqrt{\frac{2}{3}} |21\rangle + \sqrt{\frac{1}{3}} |32\rangle \right) \right]_{CDF}$$

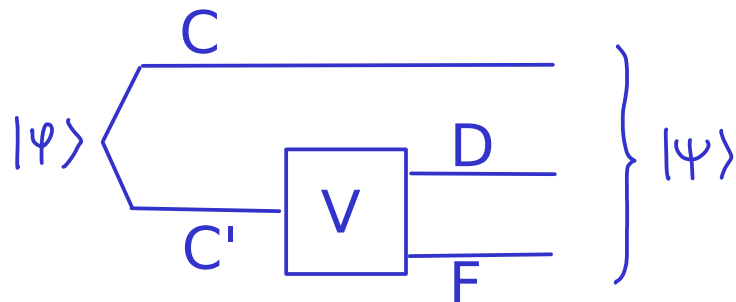
Idea: $\Psi^{CF} = \left(\frac{1}{2} |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1| \right)_C \otimes \left(\frac{2}{3} |1\rangle\langle 1| + \frac{1}{3} |2\rangle\langle 2| \right)_F$

Map $|0\rangle_D \rightarrow |00\rangle_{D_1 D_2}$, $|1\rangle_D \rightarrow |01\rangle_{D_1 D_2}$, $|2\rangle_D \rightarrow |10\rangle_{D_1 D_2}$, $|3\rangle_D \rightarrow |11\rangle_{D_1 D_2}$,

$$\begin{aligned} |\Psi\rangle &\rightarrow \frac{1}{\sqrt{2}} \left[|0\rangle \left(\sqrt{\frac{2}{3}} |001\rangle + \sqrt{\frac{1}{3}} |012\rangle \right) + |1\rangle \left(\sqrt{\frac{2}{3}} |101\rangle + \sqrt{\frac{1}{3}} |112\rangle \right) \right]_{CD_1 D_2 F} \\ &= \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right)_{CD_1} \otimes \left(\sqrt{\frac{2}{3}} |01\rangle + \sqrt{\frac{1}{3}} |12\rangle \right)_{D_2 F} \end{aligned}$$

* If tracing out D uncorrelates CF, can rotate D to uncorrelate the purification

Decoupling approach (exact case)



Shorthands: $\Psi = |\Psi\rangle\langle\Psi|$

$$\Psi^{CF} = \text{tr}_D |\Psi\rangle\langle\Psi|$$

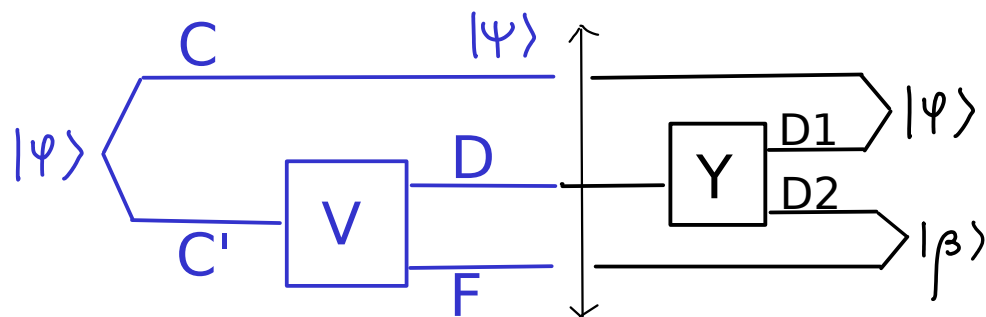
$$\Psi^F = \text{tr}_{CD} |\Psi\rangle\langle\Psi|$$

$$\Psi^C = \text{tr}_{DF} |\Psi\rangle\langle\Psi|$$

Lemma: if $\Psi^{CF} = \Psi^C \otimes \Psi^F$

then \exists isometry $Y: D \rightarrow D_1 D_2$,

$$I_{CF} \otimes Y_D |\Psi\rangle = |\Psi\rangle_{CD_1} \otimes |\beta\rangle_{D_2F} \quad \text{where } \beta^F = \Psi^F.$$

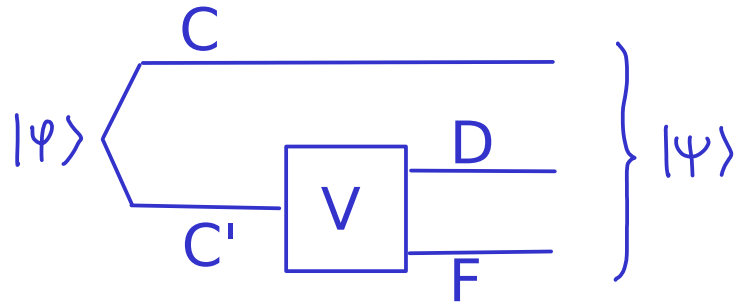


Interpretation:

C is reference to C'
channel V takes input C'
to output D & env F.

If env F is uncorrelated with reference C,
then, the output D can be decoded to
recover the state on CC' (now on CD1),
reverting the noisy quantum channel.

Decoupling approach (exact case)



Shorthands: $\Psi = |\psi\rangle\langle\psi|$

$$\Psi^{CF} = \text{tr}_D |\psi\rangle\langle\psi|$$

$$\Psi^F = \text{tr}_{CD} |\psi\rangle\langle\psi|$$

$$\Psi^C = \text{tr}_{DF} |\psi\rangle\langle\psi|$$

Lemma: if $\Psi^{CF} = \Psi^C \otimes \Psi^F$

then \exists isometry $Y: D \rightarrow D_1 D_2$,

$$I_{CF} \otimes Y_D |\psi\rangle = |\psi\rangle_{CD_1} \otimes |\beta\rangle_{D_2F} \text{ where } \beta^F = \Psi^F.$$

Proof: Consider 2 possible purifications of Ψ^{CF}

$$|\psi\rangle_{CDF} \text{ and } |\psi\rangle_{CD_1} \otimes |\beta\rangle_{D_2F}$$

any 2 purifications are related by an isometry between the purifying systems, so, $\exists Y: D \rightarrow D_1 D_2$ so that

$$I_{CF} \otimes Y_D |\psi\rangle = |\psi\rangle_{CD_1} \otimes |\beta\rangle_{D_2F}$$

Decoupling approach (approximate case via Uhlmann's thm)

Recall Uhlmann's theorem:

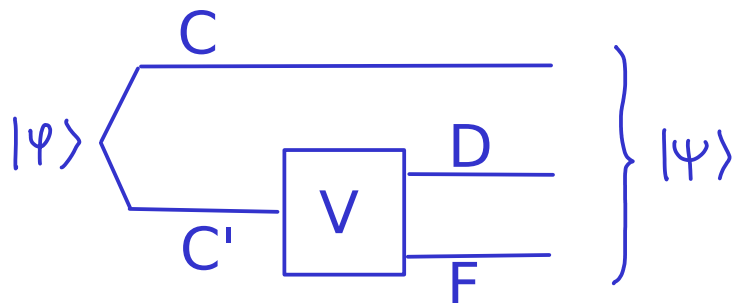
For two states ρ, σ on the same system, their fidelity

$$F(\rho, \sigma) = \max |\langle \mu | \nu \rangle|$$

where $|\nu\rangle$ is any fixed purification of σ

and the max is over all possible purifications $|\mu\rangle$ of ρ .

Decoupling approach (approximate case via Uhlmann's thm)



Lemma: if $\|\psi^{CF} - \alpha^C \otimes \beta^F\|_1 \leq \epsilon$

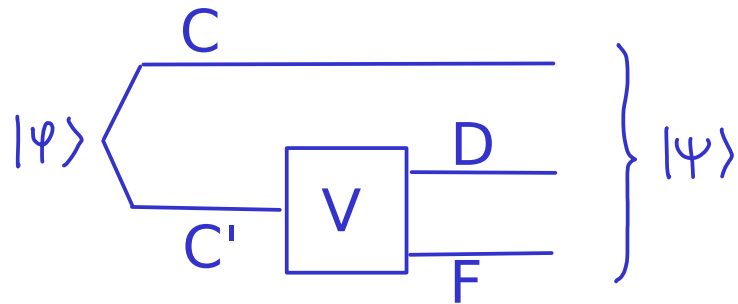
then \exists isometry $Y: D \rightarrow D_1 D_2$,

$F(I_{CF} \otimes Y_D |\psi\rangle, |\alpha\rangle_{CD_1} \otimes |\beta\rangle_{D_2 F}) \geq 1 - \frac{\epsilon}{2}$ easily converted to trace distance

Proof: $\|\psi^{CF} - \alpha^C \otimes \beta^F\|_1 \leq \epsilon \Rightarrow F(\psi^{CF}, \alpha^C \otimes \beta^F) \geq 1 - \frac{\epsilon}{2}$

Nielsen & Chuang 1st Ed (9.109)

Decoupling approach (approximate case via Uhlmann's thm)



Lemma: if $\|\psi^{CF} - \alpha^C \otimes \beta^F\|_1 \leq \epsilon$

then \exists isometry $Y: D \rightarrow D_1 D_2$,

$$F(I_{CF} \otimes Y_D |\psi\rangle, |\alpha\rangle_{CD_1} \otimes |\beta\rangle_{D_2F}) \geq 1 - \frac{\epsilon}{2}$$

Proof: $\|\psi^{CF} - \alpha^C \otimes \beta^F\|_1 \leq \epsilon \Rightarrow F(\psi^{CF}, \alpha^C \otimes \beta^F) \geq 1 - \frac{\epsilon}{2}$

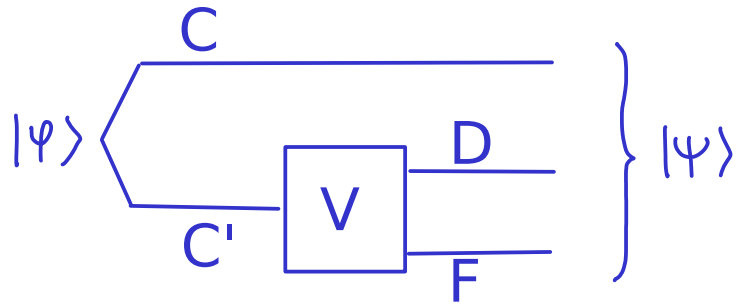
// Uhlmann's Thm

$$\max_{\text{over } |\mu\rangle} \text{that purifies } \psi^{CF} \quad \left| \langle \mu | \cdot |\alpha\rangle_{CD_1} \otimes |\beta\rangle_{D_2F} \right| \quad \leftarrow \star$$

Let $|\mu^*\rangle$ attain the max. Any 2 purifications of ψ^{CF} are related by an isometry on the purifying system. $\because |\psi\rangle, |\mu^*\rangle$ both purify ψ^{CF} ,

$\exists Y: D \rightarrow D_1 D_2, |\mu^*\rangle = I_{CF} \otimes Y_D |\psi\rangle$. Sub into \star proves the lemma.

Decoupling approach (approximate case via Uhlmann's thm)



Lemma: if $\|\psi^{CF} - \alpha^C \otimes \beta^F\|_1 \leq \epsilon$

then \exists isometry $Y: D \rightarrow D_1 D_2$,

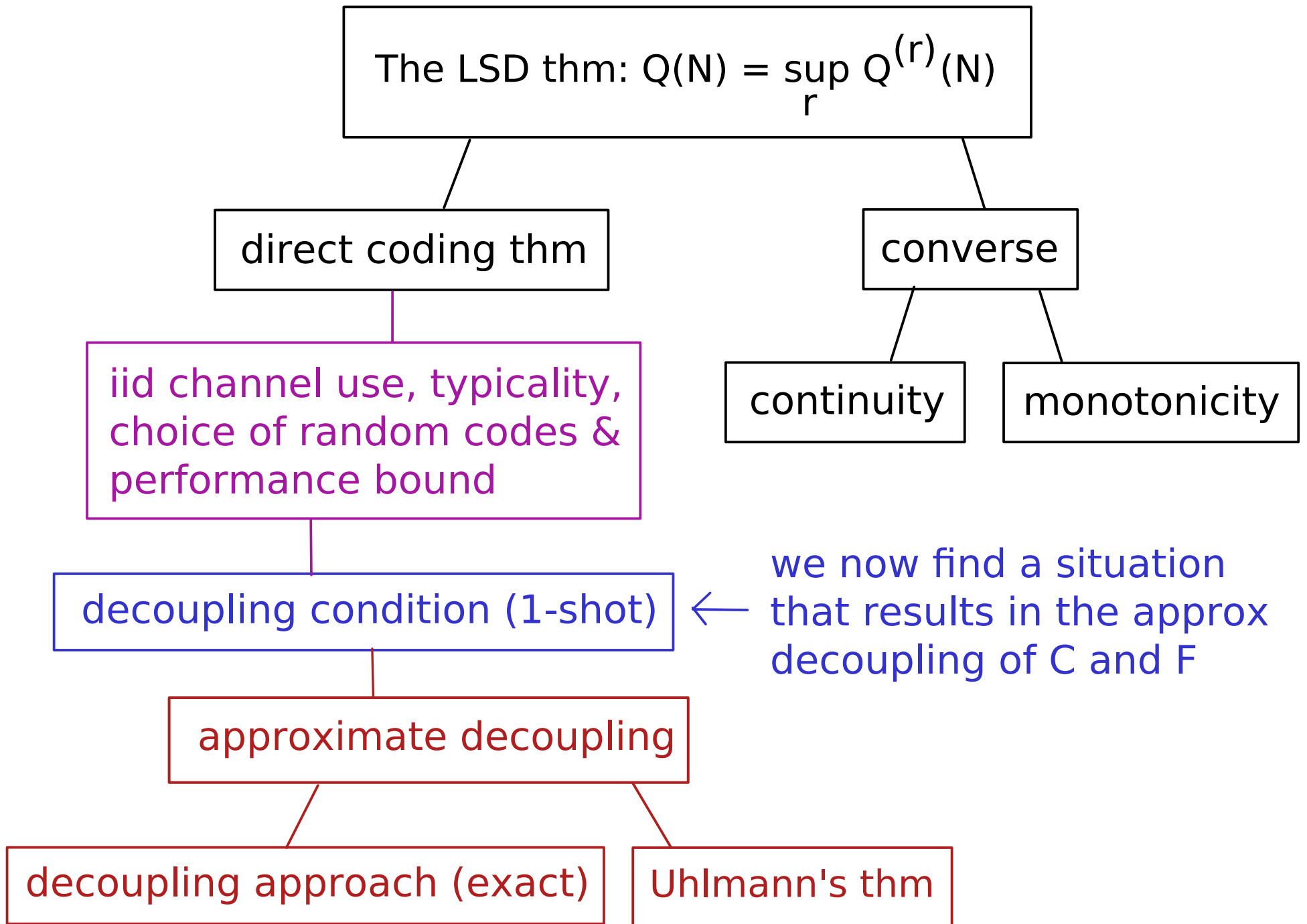
$$F(I_{CF} \otimes Y_D |\psi\rangle, |\alpha\rangle_{CD_1} \otimes |\beta\rangle_{D_2 F}) \geq 1 - \frac{\epsilon}{2}$$

Note that with the approx, we do not demand $\alpha^C = \psi^C$
 this formulation is more convenient later.

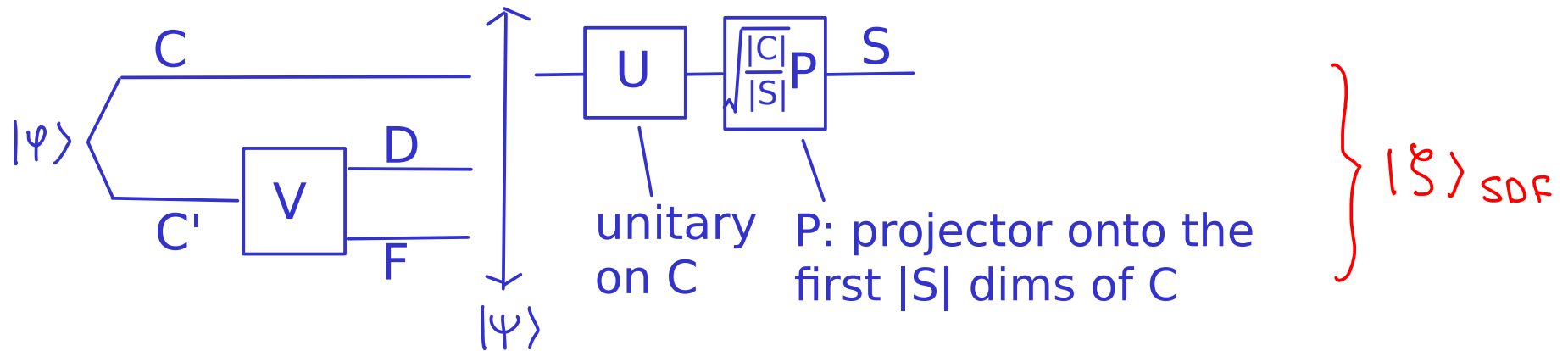
Also, the above is very similar to setting $\alpha^C = \psi^C$

since, by monotonicity of trace distance under partial tracing:

$$\|\psi^{CF} - \alpha^C \otimes \beta^F\|_1 \leq \epsilon \Rightarrow \|\psi^C - \alpha^C\|_1 \leq \epsilon$$



Decoupling condition (1-shot):



Let $|\xi\rangle = \sqrt{\frac{|C|}{|S|}} P U |\psi\rangle$ (a vector not normalized, on SDF)

Theorem: if U chosen according to the Haar measure,

$$\text{then } \mathbb{E}_U \left\| \underbrace{|\xi\rangle^{SF}}_{\text{not normalized}} - \underbrace{\pi^S \otimes \psi^F}_{\text{max mixed state on S}} \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\text{Tr}[(\psi^D)^2] \right)^{\frac{1}{2}}$$

Want $\mathbb{E}_u \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\text{Tr}[(\Psi^D)^2] \right)^{\frac{1}{2}}$

/
\

not normalized
max mixed state on S

Proof:

(1) replace 1-norm by 2-norm, using Cauchy-Schwartz ineq

$$\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2$$

Elaborate (1):

CS ineq: $\sum_i |x_i| |y_i| \leq \sqrt{\sum_i x_i^2} \sqrt{\sum_j y_j^2}$

For any hermitian $d \times d$ matrix M , let $\{x_i\}$ = eigenvalues of M
 let $y_j = 1$ for all j

Then $\|M\|_2 = \sum_i |x_i| = \sum_i |x_i| |y_i| \stackrel{CS}{\leq} \sqrt{\sum_i x_i^2} \sqrt{\sum_j y_j^2} = \|M\|_2 \sqrt{d}$

Want $\mathbb{E}_U \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\text{Tr}[(\Psi^D)^2] \right)^{\frac{1}{2}}$

not normalized

max mixed state on S

Proof:

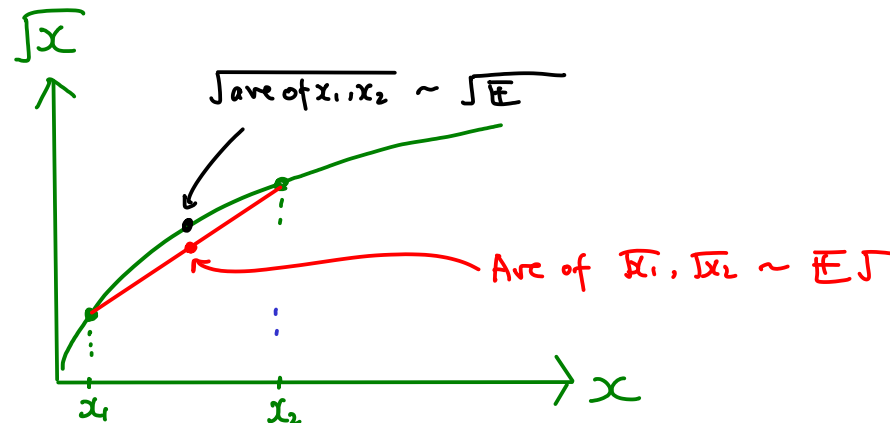
(1) replace 1-norm by 2-norm, using Cauchy-Schwartz ineq

$$\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2$$

(2) use concavity of square root

$$\mathbb{E}_U \sqrt{\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2} \leq \sqrt{\mathbb{E}_U \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2}$$

Elaborate (2):



$$\text{Want } \mathbb{E}_U \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\text{Tr}[(\psi^D)^2] \right)^{\frac{1}{2}}$$

not normalized

max mixed state on S

Proof:

(1) replace 1-norm by 2-norm, using Cauchy-Schwartz ineq

$$\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2$$

(2) use concavity of square root

$$\mathbb{E}_U \sqrt{\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2} \leq \sqrt{\mathbb{E}_U \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2}$$

(3) expand

$$\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2 = \text{Tr} \left(\xi^{SF} - \pi^S \otimes \psi^F \right) \left(\xi^{SF} - \pi^S \otimes \psi^F \right)$$

Elaborate (3):

$$\begin{aligned} \|\xi^{SF} - \pi^S \otimes \Psi^F\|_2^2 &= \text{tr} \left(\xi^{SF} - \pi^S \otimes \Psi^F \right) \left(\xi^{SF} - \pi^S \otimes \Psi^F \right) \\ &= \text{tr} \left(\xi^{SF} \right)^2 - 2 \underbrace{\text{tr} \left(\pi^S \otimes \Psi^F \right) \xi^{SF}} + \underbrace{\text{tr} \left(\pi^S \otimes \Psi^F \right)^2} \end{aligned}$$

Useful fact on partial trace:

$$\text{tr} \left(I_1 \otimes K_2 \right) M_{12} = \text{tr} K \left(\text{tr}_1 M_{12} \right)$$

\uparrow on sys 1,2 \uparrow on sys 2 \uparrow

NB $\pi^S = \frac{I}{|S|}$

$$\frac{1}{|S|} \text{tr} \Psi^F \xi^F$$

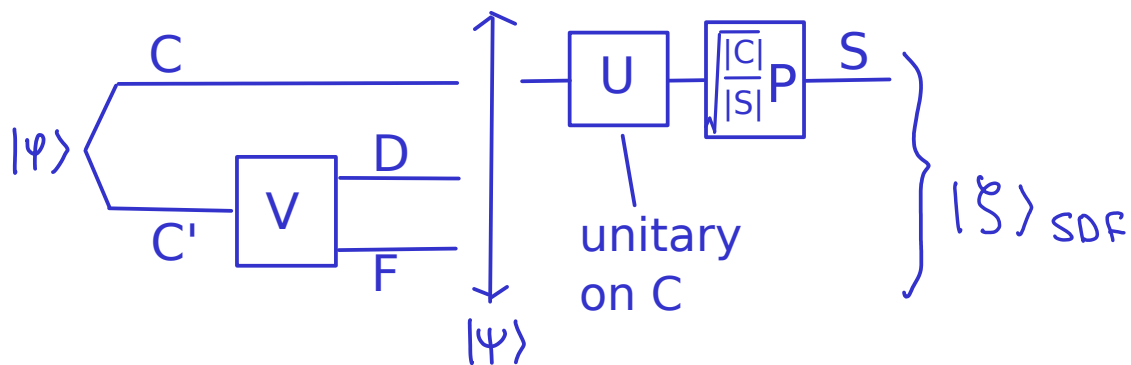
\uparrow on F \uparrow

$$\frac{1}{|S|} \text{tr} (\Psi^F)^2$$

$$\text{tr} \left[\frac{I}{|S|^2} \otimes (\Psi^F)^2 \right]$$

//

$$\frac{1}{|S|} \text{tr} (\Psi^F)^2$$



NB $\Psi^F = \xi^F$

Elaborate (3):

$$\begin{aligned}
 \|\xi^{SF} - \pi^S \otimes \Psi^F\|_2^2 &= \text{tr} \left(\xi^{SF} - \pi^S \otimes \Psi^F \right) \left(\xi^{SF} - \pi^S \otimes \Psi^F \right) \\
 &= \text{tr} \left(\xi^{SF} \right)^2 - 2 \underbrace{\text{tr} \left(\pi^S \otimes \Psi^F \right) \xi^{SF}}_{\substack{\frac{1}{|S|} \text{tr} \Psi^F \xi^F \\ \uparrow \quad \uparrow \\ \text{on } F}} + \underbrace{\text{tr} \left(\pi^S \otimes \Psi^F \right)^2}_{\substack{\text{tr} \left[\frac{I}{|S|^2} \otimes (\Psi^F)^2 \right] \\ // \\ \frac{1}{|S|} \text{tr} (\Psi^F)^2}} \\
 &= \text{tr} \left(\xi^{SF} \right)^2 - \frac{1}{|S|} \text{tr} (\Psi^F)^2
 \end{aligned}$$

Want $\mathbb{E}_u \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\text{Tr}[(\psi^D)^2] \right)^{\frac{1}{2}}$

/ not normalized \ max mixed state on S

Proof:

(1) replace 1-norm by 2-norm, using Cauchy-Schwartz ineq

$$\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2$$

(2) use concavity of square root

$$\mathbb{E}_u \sqrt{\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2} \leq \sqrt{\mathbb{E}_u \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2}$$

(3) expand $\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2 = \text{tr} \left(\xi^{SF} \right)^2 - \frac{1}{|S|} \text{tr}(\psi^F)^2$

(4) evaluate $\mathbb{E}_u \text{tr} \left(\xi^{SF} \right)^2$

Elaborate (4):

$$\mathbb{E}_u \text{tr} \left(\xi^{SF} \right)^2 = \mathbb{E}_u \text{tr}_{S_1 F_1 S_2 F_2} \left(\xi^{S_1 F_1} \otimes \xi^{S_2 F_2} \right) \sum_{S_1 F_1 S_2 F_2}$$

swapping S1 F1 with S2 F2

$$\sum_{S_1 F_1 S_2 F_2} = \sum_{S_1 S_2} \otimes \sum_{F_1 F_2}$$

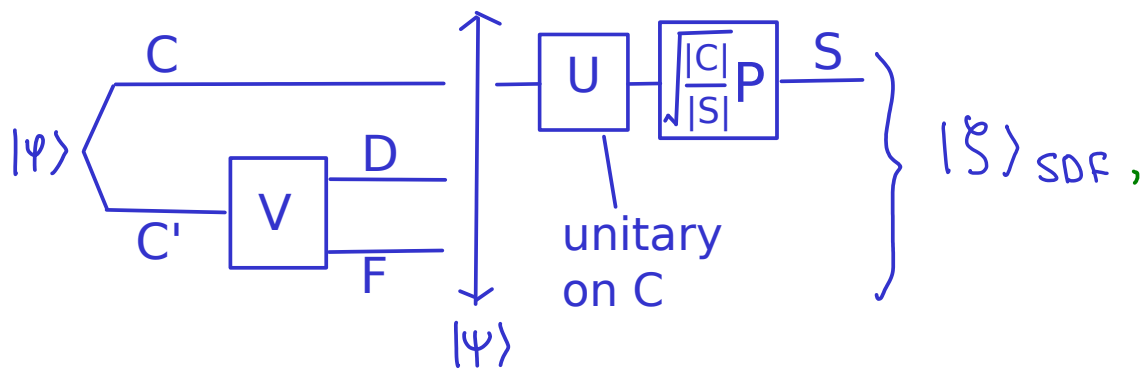
A neat trick:

$$\begin{aligned} \text{tr} \left(M \otimes N \right) \left(\sum_{ij} |ij\rangle \langle jil| \right) &= \sum_{ij} \langle jil | M \otimes N | ij \rangle \\ &= \sum_{ij} \langle j | M | i \rangle \langle i | N | j \rangle \\ &= \sum_j \sum_i \langle j | M | i \rangle \langle i | N | j \rangle \\ &= \text{tr} M N \end{aligned}$$

Elaborate (4):

$$\mathbb{E}_U \text{tr} (\xi^{SF})^2 = \mathbb{E}_U \text{tr}_{S_1 F_1 S_2 F_2} (\xi^{S_1 F_1} \otimes \xi^{S_2 F_2}) \left(\sum_{S_1 S_2} \otimes \sum_{F_1 F_2} \right)$$

$$= \mathbb{E}_U \text{tr}_{S_1 F_1 S_2 F_2} \frac{|C|^2}{|S|^2} (P U \otimes P U \otimes I_{F_1 F_2}) \psi^{C_1 F_1} \otimes \psi^{C_2 F_2} (U^T P \otimes U^T P \otimes I_{F_1 F_2}) \left(\sum_{S_1 S_2} \otimes \sum_{F_1 F_2} \right)$$



$$\therefore \xi^{SF} = \frac{|C|}{|S|} (P U \otimes I_F) \psi^{CF} (U^T P \otimes I_F)$$

$$\xi^{S_1 F_1} \otimes \xi^{S_2 F_2} = \frac{|C|^2}{|S|^2} (P U \otimes P U \otimes I_{F_1 F_2}) \psi^{C_1 F_1} \otimes \psi^{C_2 F_2} (U^T P \otimes U^T P \otimes I_{F_1 F_2})$$

Elaborate (4):

$$\mathbb{E}_U \text{tr} \left(\xi^{SF} \right)^2 = \mathbb{E}_U \text{tr}_{S_1 F_1 S_2 F_2} \left(\xi^{S_1 F_1} \otimes \xi^{S_2 F_2} \right) \left(\sum S_1 S_2 \otimes \sum F_1 F_2 \right)$$

$$= \mathbb{E}_U \text{tr}_{S_1 F_1 S_2 F_2} \frac{|c|^2}{|s|^2} \left(P U \otimes P U \otimes I_{F_1 F_2} \right) \Psi^{C_1 F_1} \otimes \Psi^{C_2 F_2} \left(U^T P \otimes U^T P \otimes I_{F_1 F_2} \right) \left(\sum S_1 S_2 \otimes \sum F_1 F_2 \right)$$

cyclic prop of trace

$$= \mathbb{E}_U \text{tr}_{S_1 F_1 S_2 F_2} \frac{|c|^2}{|s|^2} \Psi^{C_1 F_1} \otimes \Psi^{C_2 F_2} \left(U^T P \otimes U^T P \otimes I_{F_1 F_2} \right) \left(\sum S_1 S_2 \otimes \sum F_1 F_2 \right) \left(P U \otimes P U \otimes I_{F_1 F_2} \right)$$

E & tr commute Ψ indep of U

multiply and simplify

$$= \text{tr}_{C_1 F_1 C_2 F_2} \frac{|c|^2}{|s|^2} \Psi^{C_1 F_1} \otimes \Psi^{C_2 F_2} \underbrace{\mathbb{E}_U \left[\left(U^T P \otimes U^T P \right) \sum S_1 S_2 \left(P U \otimes P U \right) \right]}_{\text{multiply and simplify}} \otimes \sum F_1 F_2$$

(see 0702005,
or 1501.04592,
or Watrous bk)

$$\left(\frac{|s|}{|c|} \right)^2 \left(\frac{1 - \frac{1}{|c||s|}}{1 - \frac{1}{|c|^2}} \right) \sum^{C_1 C_2} + \frac{|s|}{|c|^2} \left(\frac{1 - \frac{|s|}{|c|}}{1 - \frac{1}{|c|^2}} \right) I^{C_1 C_2}$$

$$= \text{tr}_{C_1 F_1 C_2 F_2} \Psi^{C_1 F_1} \otimes \Psi^{C_2 F_2} \left[\left(\frac{1 - \frac{1}{|c||s|}}{1 - \frac{1}{|c|^2}} \right) \sum^{C_1 C_2} + \frac{1}{|s|} \left(\frac{1 - \frac{|s|}{|c|}}{1 - \frac{1}{|c|^2}} \right) I^{C_1 C_2} \right] \otimes \sum F_1 F_2$$

$$= \text{tr}_{C_1 F_1 C_2 F_2} \psi^{C_1 F_1} \otimes \psi^{C_2 F_2} \left[\left(\frac{1 - \frac{1}{|c||s|}}{1 - \frac{1}{|c|^2}} \right) \sum_{C_1 C_2} + \frac{1}{|s|} \left(\frac{1 - \frac{|s|}{|c|}}{1 - \frac{1}{|c|^2}} \right) \mathbb{I}^{C_1 C_2} \right] \otimes \sum_{F_1 F_2}$$

$$= \left(\frac{1 - \frac{1}{|c||s|}}{1 - \frac{1}{|c|^2}} \right) \text{tr}_{C_1 F_1 C_2 F_2} \psi^{C_1 F_1} \otimes \psi^{C_2 F_2} \sum_{C_1 C_2} \otimes \sum_{F_1 F_2}$$

neat trick / \swarrow

$$+ \frac{1}{|s|} \left(\frac{1 - \frac{|s|}{|c|}}{1 - \frac{1}{|c|^2}} \right) \text{tr}_{C_1 F_1 C_2 F_2} \psi^{C_1 F_1} \otimes \psi^{C_2 F_2} \mathbb{I}^{C_1 C_2} \otimes \sum_{F_1 F_2}$$

\searrow partial trace

$$= \underbrace{\left(\frac{1 - \frac{1}{|c||s|}}{1 - \frac{1}{|c|^2}} \right)}_{\leq 1} \text{tr}_{CF} (\psi^{CF})^2 + \underbrace{\frac{1}{|s|} \left(\frac{1 - \frac{|s|}{|c|}}{1 - \frac{1}{|c|^2}} \right)}_{\leq 1} \underbrace{\text{tr}_{F_1 F_2} \left(\underbrace{\text{tr}_{C_1 C_2} \psi^{C_1 F_1} \otimes \psi^{C_2 F_2}}_{\psi^{F_1} \otimes \psi^{F_2}} \right)}_{\text{neat trick}} \sum_{F_1 F_2}$$

$$\text{tr} (\psi^F)^2$$

$$\leq \text{tr}_{CF} (\psi^{CF})^2 + \frac{1}{|s|} \text{tr} (\psi^F)^2$$

Want $\mathbb{E}_u \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\text{Tr}[(\psi^D)^2] \right)^{\frac{1}{2}}$

got:

(1) replace 1-norm by 2-norm, using Cauchy-Schwartz ineq

$$\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2$$

(2) use concavity of square root

$$\mathbb{E}_u \sqrt{\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2} \leq \sqrt{\mathbb{E}_u \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2}$$

(3) expand $\left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2 = \text{tr} \left(\xi^{SF} \right)^2 - \frac{1}{|S|} \text{tr} (\psi^F)^2$

(4) evaluate $\mathbb{E}_u \text{tr} \left(\xi^{SF} \right)^2 \leq \text{tr}_{CF} \left(\psi^{CF} \right)^2 + \frac{1}{|S|} \text{tr} (\psi^F)^2$

together:

$$\mathbb{E}_u \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \stackrel{(1)}{\leq} \mathbb{E}_u |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2$$

$$\stackrel{(2)}{\leq} |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \sqrt{\mathbb{E}_u \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_2^2}$$

(3) expand $\| \xi^{SF} - \pi^S \otimes \psi^F \|_2^2 = \text{tr} \left(\xi^{SF} \right)^2 - \frac{1}{|S|} \text{tr} (\psi^F)^2$

(4) evaluate $\mathbb{E}_U \text{tr} \left(\xi^{SF} \right)^2 \leq \text{tr}_{CF} (\psi^{CF})^2 + \frac{1}{|S|} \text{tr} (\psi^F)^2$

from (3)&(4):

$$\begin{aligned} \mathbb{E}_U \| \xi^{SF} - \pi^S \otimes \psi^F \|_2^2 &= \mathbb{E}_U \text{tr} \left(\xi^{SF} \right)^2 - \frac{1}{|S|} \text{tr} (\psi^F)^2 \\ &\leq \text{tr}_{CF} (\psi^{CF})^2 + \frac{1}{|S|} \text{tr} (\psi^F)^2 - \frac{1}{|S|} \text{tr} (\psi^F)^2 \\ &= \text{tr}_{CF} (\psi^{CF})^2 = \left(\text{Tr} [(\psi^D)^2] \right)^{\frac{1}{2}} \end{aligned}$$

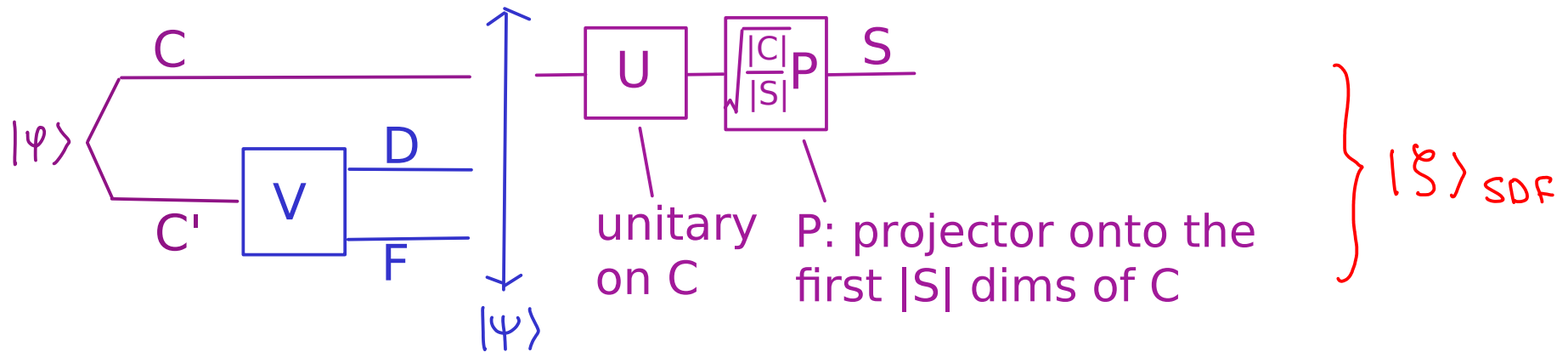
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↙

returning to previous page:

$$\mathbb{E}_U \| \xi^{SF} - \pi^S \otimes \psi^F \|_1 \stackrel{(1)}{\leq} \mathbb{E}_U |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \| \xi^{SF} - \pi^S \otimes \psi^F \|_2$$

$$\stackrel{(2)}{\leq} |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \sqrt{\mathbb{E}_U \| \xi^{SF} - \pi^S \otimes \psi^F \|_2^2} \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\text{Tr} [(\psi^D)^2] \right)^{\frac{1}{2}} \quad \square$$

Decoupling condition (1-shot):



Let $|\xi\rangle = \frac{|C|}{|S|} P U |\psi\rangle$ (a vector not normalized, on SDF)

Theorem: if U chosen according to the Haar measure,

$$\text{then } \mathbb{E}_U \left\| \underbrace{|\xi\rangle^{SF}}_{\text{not normalized}} - \underbrace{\pi^S \otimes \psi^F}_{\text{max mixed state on S}} \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\text{Tr}[(\psi^D)^2] \right)^{\frac{1}{2}}$$

Purple part generates a random code for channel V , code dim $|S|$. If the RHS of Theorem is small, using decoupling (approx), there exists U (or a code) s.t. V can be inverted by some Y on D .

Thur: make RHS of theorem small in (M,n) code for LSD thm.