CO781 / QIC 890:

Theory of Quantum Communication

Topic 5, part 5

Consequences of the LSD theorem

-- so what IS the quantum capacity of a quantum channel?

* what we know (degradable channels, e.g., erasure channel)

Tue

-- bounds (continuity, 1-shot)

Today

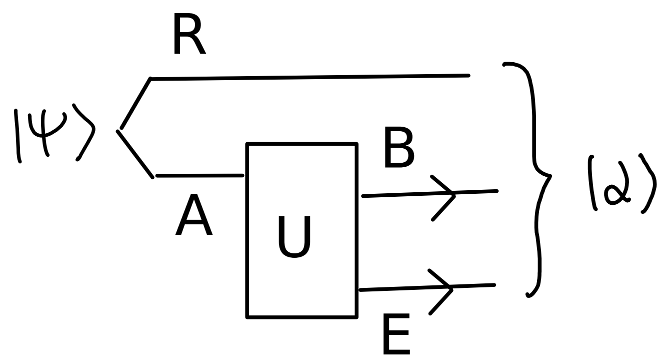
* what we know we don't know
  (nonadditivity of coherent info -- depolarizing channel)

<u>Recall:</u>

$$Q^{(1)}(N) := \max_{|\psi\rangle} I_c(R\rangle B)_{I\otimes N(|\psi\rangle\langle\psi|)_{RB}}$$

$$= \max_{|\psi\rangle} \left(S_B - S_{RB}\right)_{I\otimes N(|\psi\rangle\langle\psi|)_{RB}}$$

$$= \max_{|\psi\rangle} \left(S_B - S_E\right)_{|\alpha\rangle}$$



$$Q^{(r)}(N) := \frac{1}{r} Q^{(1)}(N^{\otimes r})$$

LSD thm: $Q(N) = \sup_r Q^{(r)}(N)$

How to evaluate the coherent information for any arbitrary channel?

Example: mixed Pauli channel (on a qubit)

$$\mathcal{N}_{\vec{q}}(\rho) = q_0 \rho + q_1 X \rho X + q_2 Y \rho Y + q_3 Z \rho Z$$

where $0 \le q_i$, $\sum_{i=0}^{3} q_i = 1$, WLOG $q_0 \geq q_1 \geq q_2 \geq q_3$

If q2=0 (a mixture of at most 2 Pauli's & equiv to dephasing channel)

then the channel is always degradable.

If q2>0 (a mixture of at least 3 Pauli's)

then the channel is NOT degradable.

One simple reason:

Recall antidegradable channel has no capacity.

When q2>0, $Q^{(1)}(\mathcal{N}_{\vec{q}}^{c}) > 0$ so $\mathcal{N}_{\vec{q}}$ cannot be degradable.

(L,Watrous 1510.01366)

Example: mixed Pauli channel (on a qubit)

$\mathcal{N}_{\vec{q}}(\rho) = q_0 \rho + q_1 X\rho X + q_2 Y\rho Y + q_3 Z\rho Z$ input A, output B

Consider the input $|\Psi_0\rangle_{RA} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Let $|\Psi_1\rangle = I\otimes X|\Psi_0\rangle$, $|\Psi_2\rangle = I\otimes Y|\Psi_0\rangle$, $|\Psi_3\rangle = I\otimes Z|\Psi_0\rangle$

Bell basis

$I\otimes\mathcal{N}_{\vec{q}}(|\Psi_0\rangle\langle\Psi_0|) = \sum_i q_i |\Psi_i\rangle\langle\Psi_i|$ so $S(RB) = H(\vec{q})$.

$\mathrm{tr}_R\, I\otimes\mathcal{N}_{\vec{q}}(|\Psi_0\rangle\langle\Psi_0|) = \frac{I}{2}$, $\therefore$ $S(B) = 1$.

So, $Q^{(1)}(\mathcal{N}_{\vec{q}}) = \max_{|\Psi\rangle} I(R\rangle B)_{I\otimes\mathcal{N}_{\vec{q}}(|\Psi\rangle\langle\Psi|)}$

$$\geq \begin{cases} I(R\rangle B)_{I\otimes\mathcal{N}_{\vec{q}}(|\Psi_0\rangle\langle\Psi_0|)} = 1 - H(\vec{q}) & \text{if } H(\vec{q}) \leq 1 \\ I(R\rangle B)_{I\otimes\mathcal{N}_{\vec{q}}(|00\rangle\langle 00|)} = 0 & \text{otherwise} \end{cases}$$

Extensive numerics and continuity results strongly indicates that $|\Psi_0\rangle$ is optimal for coherent info, but no analytical proof is known.

Special case: depolarizing channel (on a qubit)

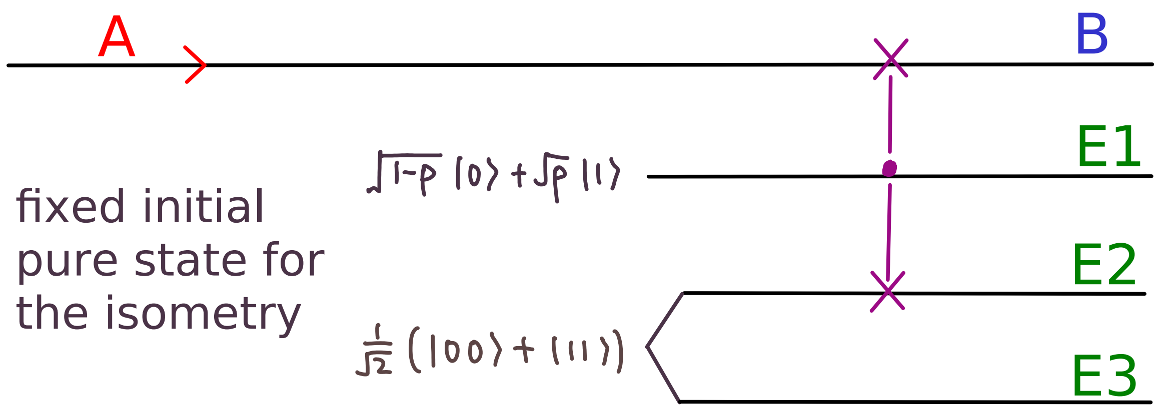
$$\mathcal{N}_q(\rho) = (1-q)\,\rho + \frac{q}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right)$$ input A, output B

$$= \left(1-\tfrac{4}{3}q\right)\rho + 4\,\frac{q}{3}\left(X\rho X + Y\rho Y + Z\rho Z + \rho\right)\frac{1}{4}$$

$$= (1-p)\,\rho + p\,\frac{I}{2}$$ where $p = \frac{4q}{3}$

Isometric extension:

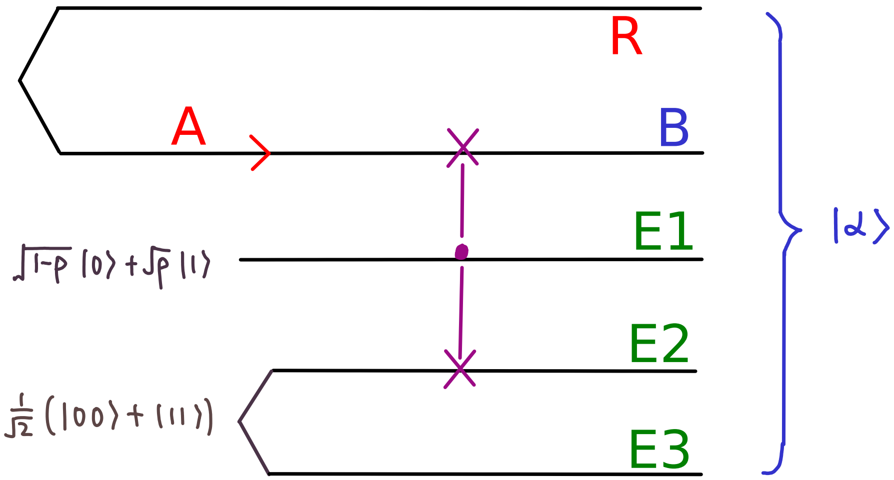
if • in |1>, swap the sys labelled by x's



We now use the symmetry of the depolarizing channel to prove that $|\psi_0\rangle$ is optimal.

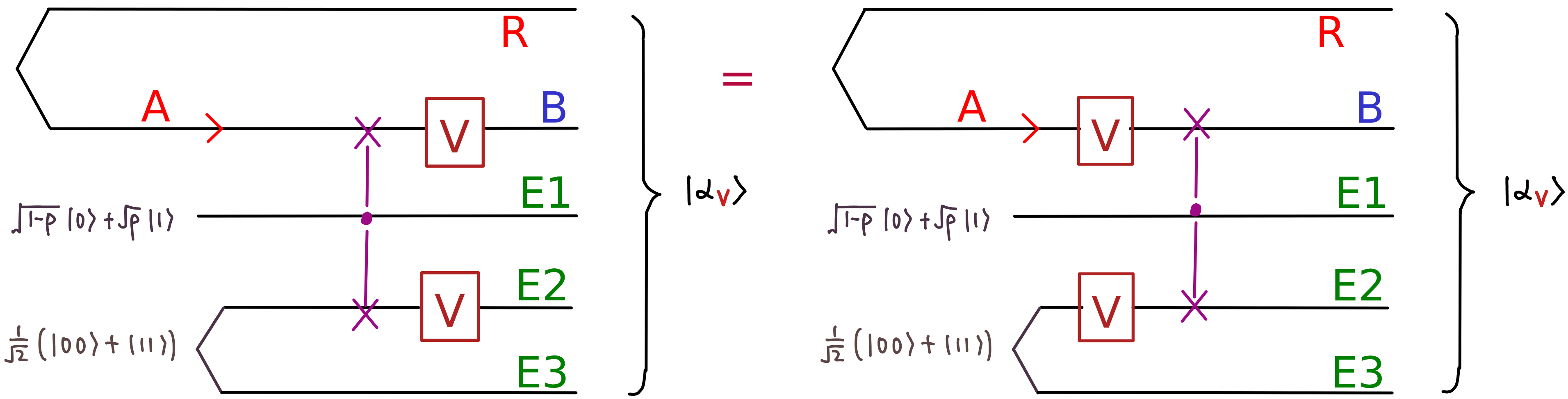
Let $|\psi\rangle_{RA}$ an optimal state for the 1-shot coherent info, so, by def,

$$Q^{(1)}(\mathcal{N}_\xi) = I(R\rangle B)_{I\otimes\mathcal{N}_\xi(|\psi\rangle\langle\psi|)} = S_B - S_E\,{}_{|\alpha\rangle\langle\alpha|}$$



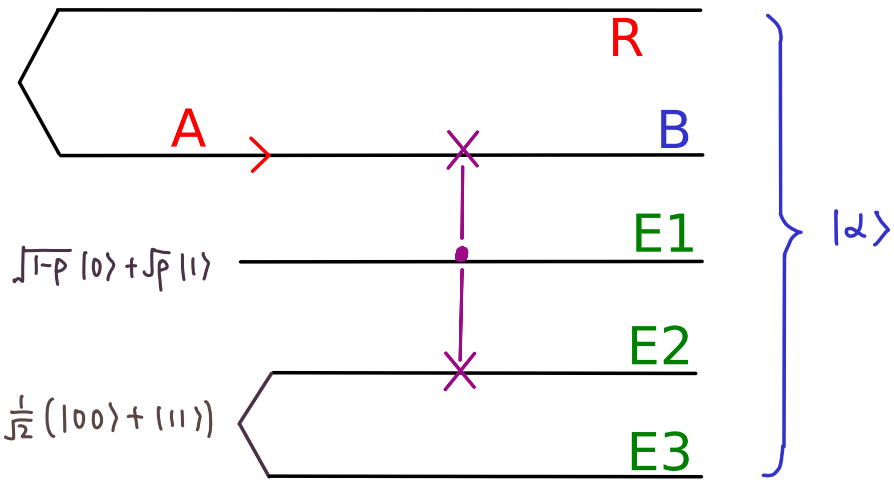
$$= S_B - S_E\,{}_{|\alpha_V\rangle\langle\alpha_V|}$$

(local unitaries on B, E don't change coh info)



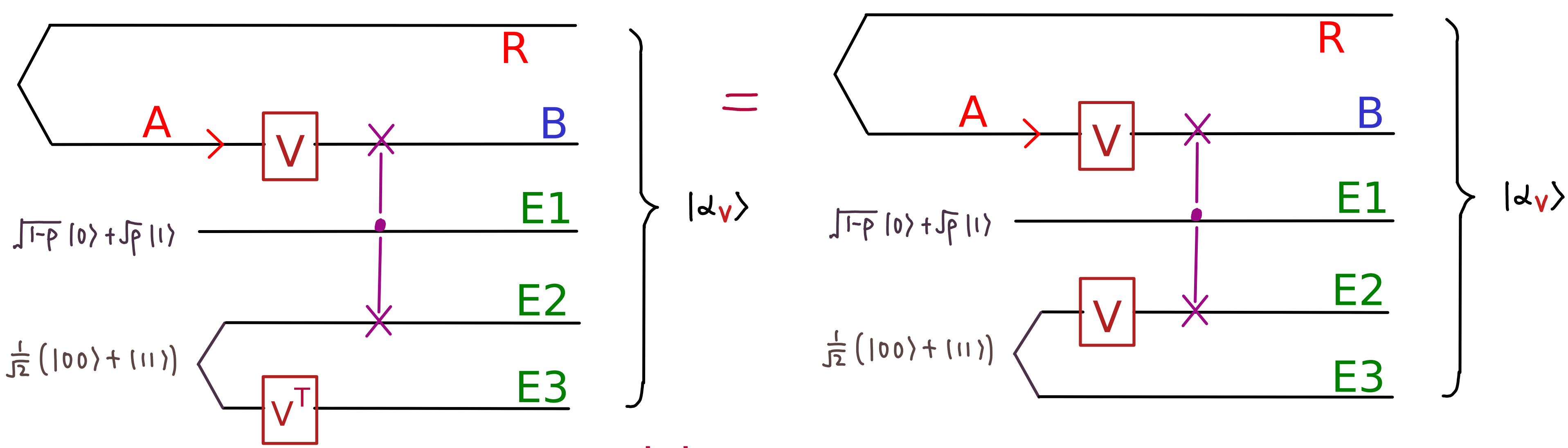
Let $|\psi\rangle_{RA}$ an optimal state for the 1-shot coherent info, so, by def,

$$Q^{(1)}(\mathcal{N}_\xi) = I(R\rangle B)_{I\otimes\mathcal{N}_\xi(|\psi\rangle\langle\psi|)} = S_B - S_E\,_{|\alpha\rangle\langle\alpha|}$$



$$= S_B - S_E\,_{|\alpha_V\rangle\langle\alpha_V|}$$

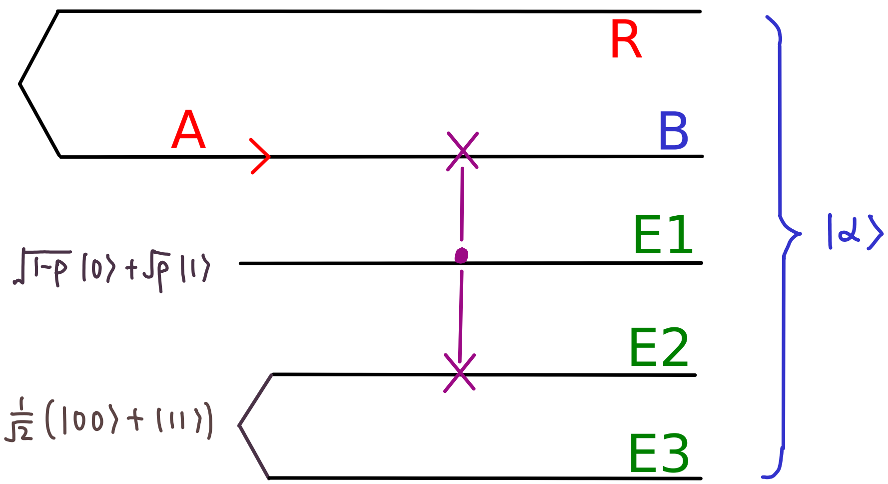
(local unitaries don't change coh info)



transpose trick

Let $|\psi\rangle_{RA}$ an optimal state for the 1-shot coherent info, so, by def,

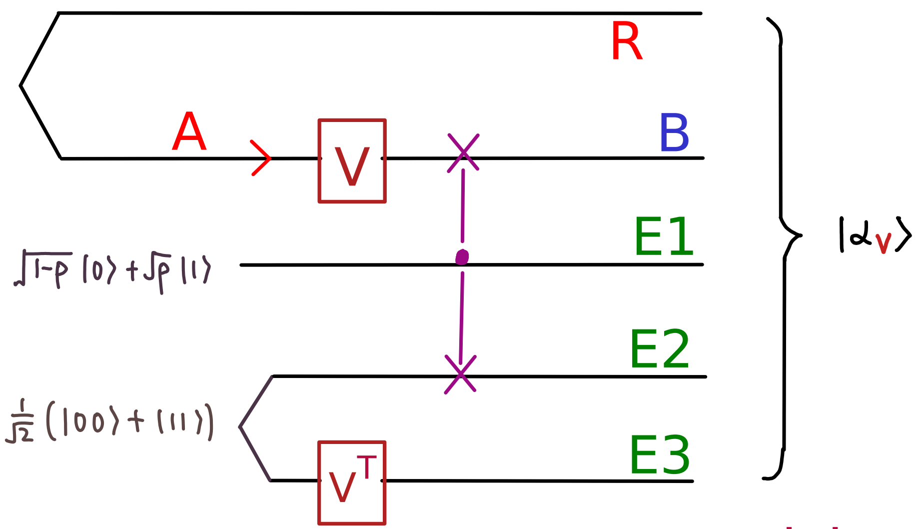
$$Q^{(1)}(\mathcal{N}_\xi) = I(R\rangle B)_{I\otimes\mathcal{N}_\xi(|\psi\rangle\langle\psi|)} = S_B - S_E\Big|_{|\alpha\rangle\langle\alpha|}$$



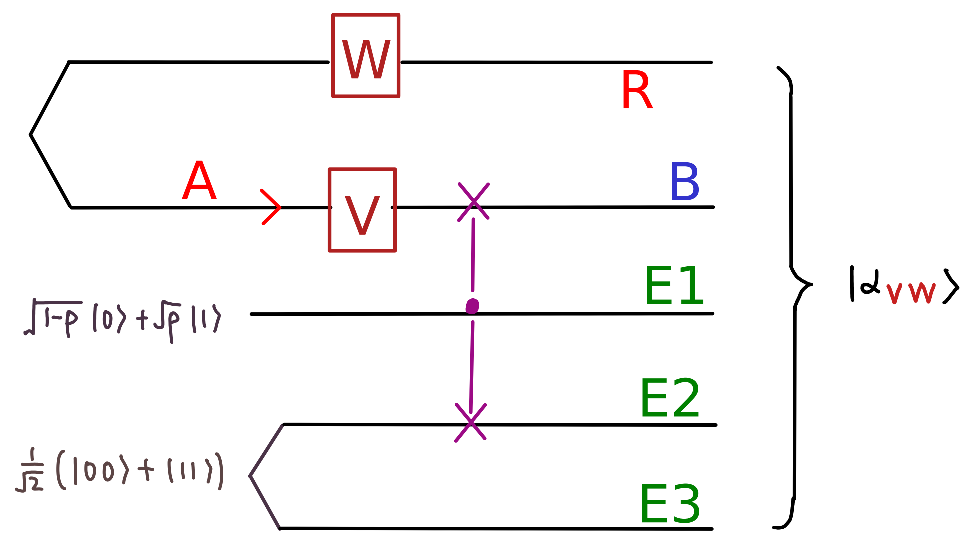
$$= S_B - S_E\Big|_{|\alpha_V\rangle\langle\alpha_V|}$$ (local unitaries don't change coh info)

$$= S_B - S_E\Big|_{|\alpha_{VW}\rangle\langle\alpha_{VW}|}$$ (local unitaries don't change coh info)

so, $W \otimes V\ |\psi\rangle_{RA}$ is also optimal



transpose trick



drop V^T and add W from LHS

Let $|\psi\rangle_{RA}$ an optimal state for the 1-shot coherent info, so, by def,

$$Q^{(1)}(N_\xi) = I(R\rangle B)_{I\otimes N_\xi(|\psi\rangle\langle\psi|)} = S_B - S_E\,{}_{|\alpha\rangle\langle\alpha|}$$

$$= S_B - S_E\,{}_{|\alpha_v\rangle\langle\alpha_v|}$$ (local unitaries don't change coh info)

$$= S_B - S_E\,{}_{|\alpha_{vw}\rangle\langle\alpha_{vw}|}$$ (local unitaries don't change coh info)

so, $W \otimes V\, |\psi\rangle_{RA}$ is also optimal

If $|\psi\rangle_{RA}$ has Schmidt decomposition $\sqrt{1-\alpha}\,|e_0\rangle|f_0\rangle + \sqrt{\alpha}\,|e_1\rangle|f_1\rangle$

choose W to rotate $|e_i\rangle$ to $|i\rangle$, V to rotate $|f_i\rangle$ to $|i\rangle$

so the optimal state $W \otimes V\, |\psi\rangle_{RA} = \sqrt{1-\alpha}\,|0\rangle|0\rangle + \sqrt{\alpha}\,|1\rangle|1\rangle = |\phi\rangle$

Ex: $$I\otimes N_\xi(|\phi\rangle\langle\phi|)_{RB} = (1-p)\begin{bmatrix} 1-\alpha & 0 & 0 & \sqrt{1-\alpha}\sqrt{\alpha} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-\alpha}\sqrt{\alpha} & 0 & 0 & \alpha \end{bmatrix} + p\begin{bmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{bmatrix} \otimes \frac{I}{2}$$

$$I\otimes N_\xi(|\phi\rangle\langle\phi|)_B = (1-p)\begin{bmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{bmatrix} + p\,\frac{I}{2}$$

so S(B) - S(RB) = $f(\alpha)$

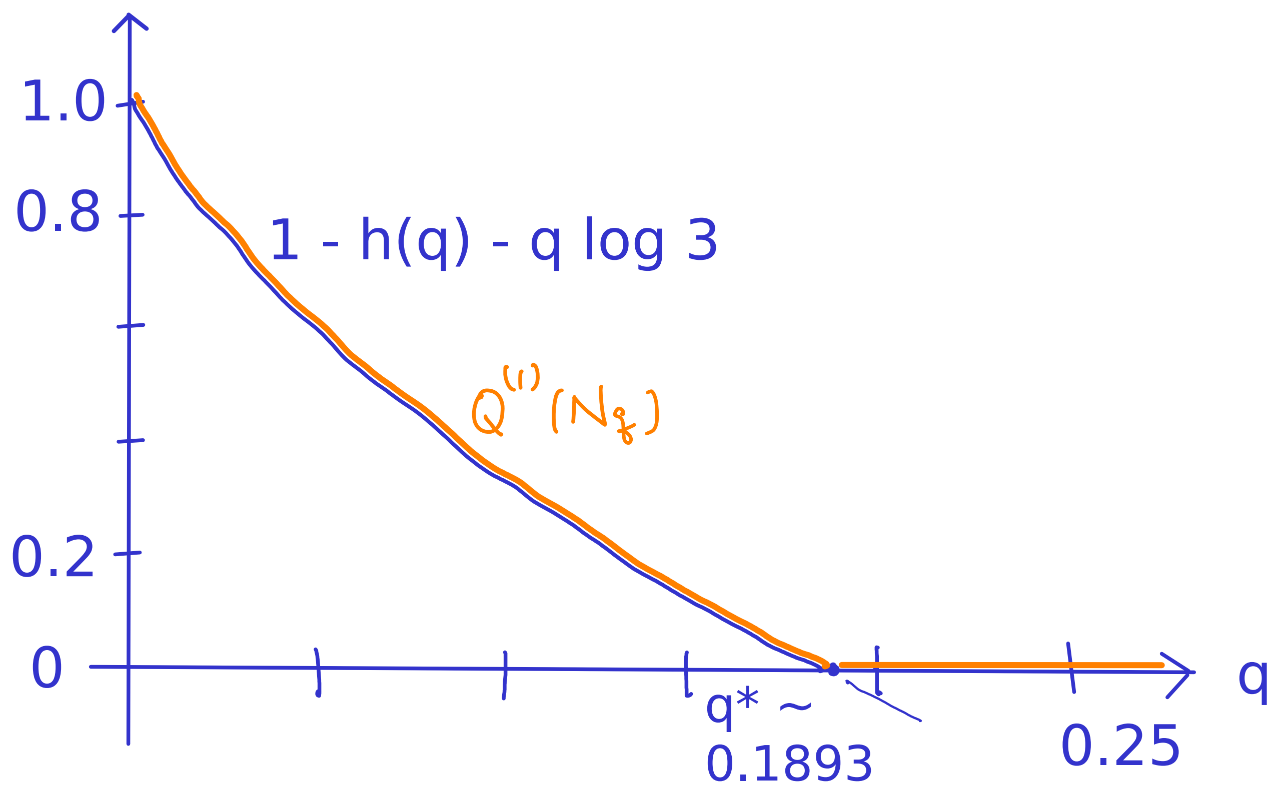
bruteforce calculus gives max at $\alpha = \frac{1}{2}$.

Special case: depolarizing channel (on a qubit)

$$\mathcal{N}_q(\rho) = (1-q)\,\rho + \frac{q}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right)$$ input A, output B

$$Q^{(1)}(\mathcal{N}_q) = \begin{cases} I(R\rangle B)_{I\otimes\mathcal{N}_q(|\psi_0\rangle\langle\psi_0|)} = 1 - H(\vec{q}) & \text{if } H(\vec{q}) \leq 1 \\ I(R\rangle B)_{I\otimes\mathcal{N}_q(|00\rangle\langle 00|)} = 0 & \text{otherwise} \end{cases}$$

where $\vec{q} = (1-q,\ q/3,\ q/3,\ q/3)$, $H(\vec{q}) = h(q) + q\log 3$



Summary so far:

For mixed Pauli channels, the MES gives a lower bound on the 1-shot coherent info (conjectured optimal),

For depolarizing channel, the MES can be proved optimal, so, we know the 1-shot information.

Next: pick $q > q^*$, show that 3-shot coherent info is positive.

Theorem: the coherent information is non-additive in general.
In particular, for some q,

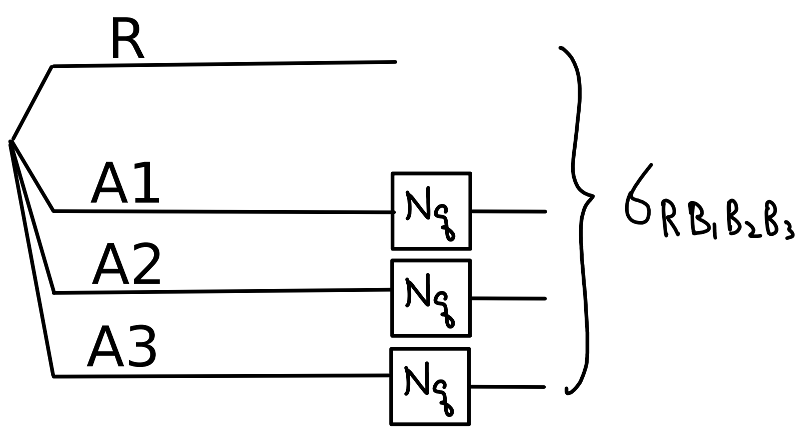
$$Q^{(1)}(N_q) = 0, \quad Q^{(3)}(N_q) > 0$$

Proof sketch:

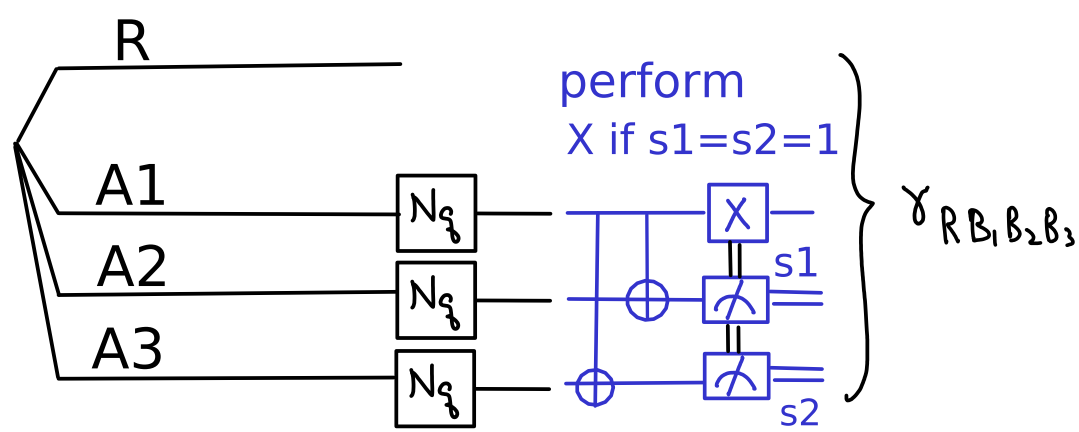
(1) we pick q > 0.18931 (when q = 0.18931, $H(\vec{q}) \doteq 1.0001$)

(2) pick the 3-use input $\frac{1}{\sqrt{2}}\left[|0\rangle_R\,|000\rangle_{A_1A_2A_3} + |1\rangle_R\,|111\rangle_{A_1A_2A_3}\right]$

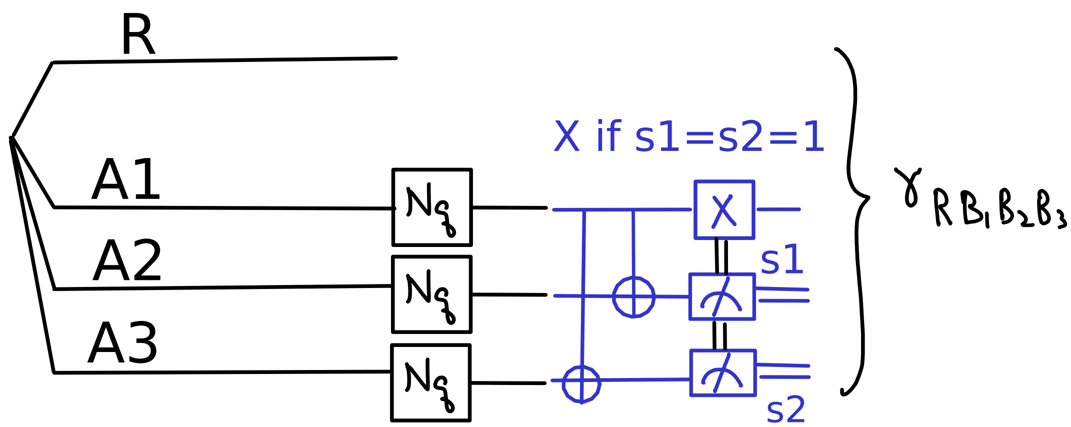
(3) the coh info is evaluated on



(4) operations on Bob's side cannot increase coh info, so, coh info of $\gamma_{RB_1B_2B_3}$ lower bounds that of σ.



(4) operations on Bob's side cannot increase coh info, so, coh info of $\gamma_{RB_1B_2B_3}$ lower bounds that of σ.

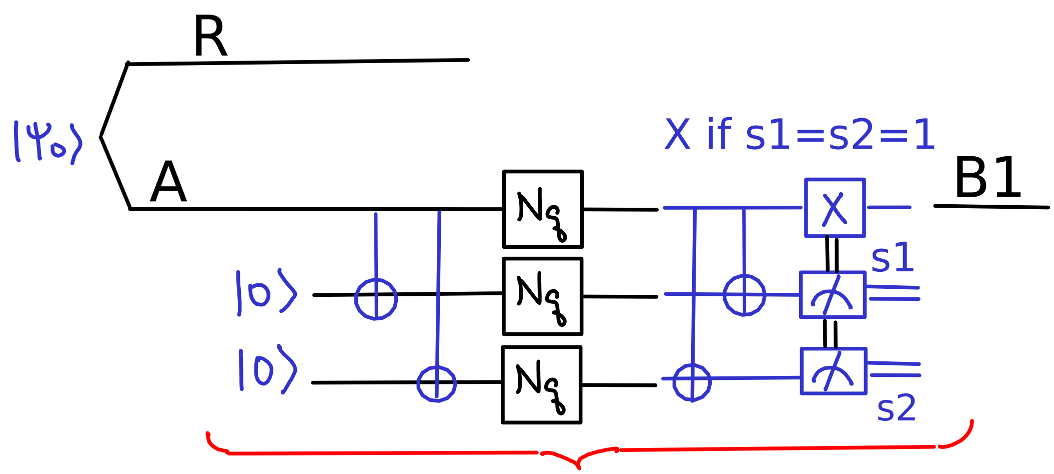


$\frac{1}{\sqrt{2}}\left[|0\rangle|000\rangle + |1\rangle|111\rangle\right]_{RA_1A_2A_3}$

(6) Since Bob has a classical system (B2B3), coh info is weighted average over s1 s2.

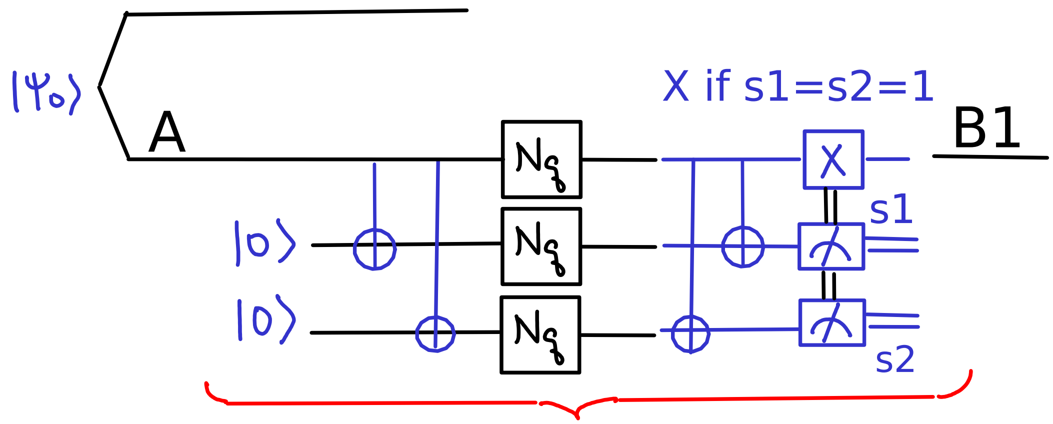
$$3Q^{(3)}(N_{\vec{q}}) \geq I(R\rangle B_1B_2B_3)_\gamma = \sum_{s_1s_2} p(s_1s_2)\, \underbrace{I(R\rangle B_1)_{\gamma_{s_1s_2}}}_{1-H(\vec{q}_{s_1s_2})}$$

(5) rephrase γ:



show for each s1 s2, this is equal to some $N_{\vec{q}_{s_1s_2}}$ so $I_c(R\rangle B_1) = 1-H(\vec{q}_{s_1s_2})$

Detail for (5)



claim: some $\mathcal{N}_{\vec{q}_{s_1 s_2}}$

$$\mathcal{N}_q(\rho) = (1-q)\,\rho + \frac{q}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right)$$

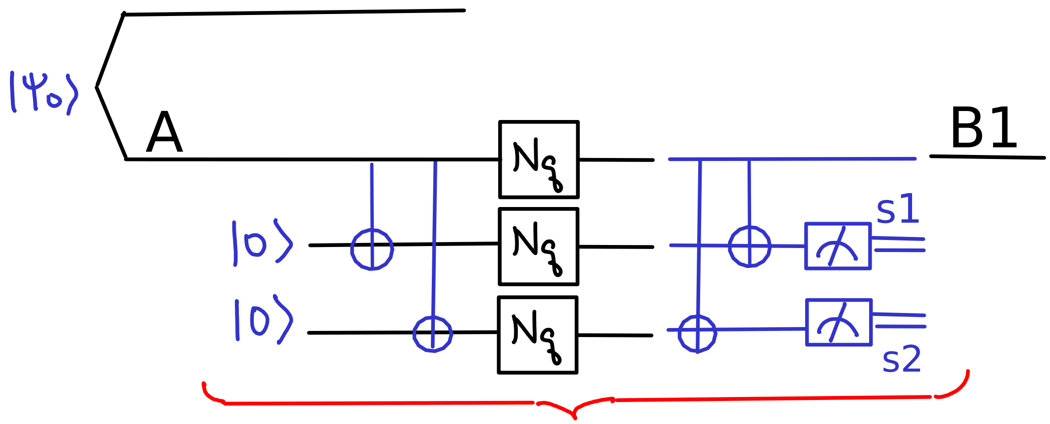
$\mathcal{N}_q(\rho)^{\otimes 3}$ has 64 Kraus operators, III, IIX, IXI, ..., ZZZ (all 3-qubit Pauli's)

Prob for each Kraus operator = $(1\text{-}q)^{3\text{-}w}\left(\frac{q}{3}\right)^{w}$ where w = # non-id Pauli's

s1 s2 is called the "syndrome."

For each syndrome there are 16 Kraus operators that can give rise to it. They further partition into 4 groups of 4, each group effects an overall Pauli from A to B1.

Detail for (5)



claim: some $\mathcal{N}_{\vec{q}_{s_1 s_2}}$

Pauli's giving s1s2=00				net effect from A to B1	prob
III	ZZI	IZZ	ZIZ	I	(1-q)^3 + 3 * (1-q)*(q/3)^2
ZII	IZI	IIZ	ZZZ	Z	3 * (1-q)^2*(q/3) + (q/3)^3
XXX	YYX	XYY	YXY	X	4 * (q/3)^3
YXX	XYX	XXY	YYY	Y	4 * (q/3)^3

sum of these 4 probs is p(s1 s2 = 00)

dividing these 4 probs by p(00) gives $\vec{q}_{00}$

Pauli's giving s1s2=10

IXI	ZYI	IYZ	ZXZ
ZXI	IYI	IXZ	ZYZ
XIX	YZX	XZY	YIY
YIX	XZX	XIY	YZY

net effect from A to B1	prob
I	(1-q)^2*(q/3) + 2 * (1-q)*(q/3)^2 + (q/3)^3
Z	(1-q)^2*(q/3) + 2 * (1-q)*(q/3)^2 + (q/3)^3
X	2 * (q/3)^3
Y	2 * (q/3)^3

sum of these 4 probs is p(s1 s2 = 10)

dividing these 4 probs by p(10) gives $\vec{q}_{10}$

Pauli's giving s1s2=01

IIX	ZIY	IZY	ZZX
ZIX	IIY	IZX	ZZY
XXI	YXZ	XYZ	YYI
YXI	XXZ	XYI	YYZ

net effect from A to B1	prob (same as above!)
I	(1-q)^2*(q/3) + 2 * (1-q)*(q/3)^2 + (q/3)^3
Z	(1-q)^2*(q/3) + 2 * (1-q)*(q/3)^2 + (q/3)^3
X	2 * (q/3)^3
Y	2 * (q/3)^3

sum of these 4 probs is p(s1 s2 = 01)

dividing these 4 probs by p(01) gives $\vec{q}_{01}$

Pauli's giving s1s2=10				net effect from A to B1	prob (same as above!)
XII	YZI	YIZ	XZZ	I	(1-q)^2*(q/3) + 2 * (1-q)*(q/3)^2 + (q/3)^3
XZI	YII	XIZ	YZZ	Z	(1-q)^2*(q/3) + 2 * (1-q)*(q/3)^2 + (q/3)^3
IXX	ZYX	ZXY	IYY	X	2 * (q/3)^3
IYX	ZXX	IXY	ZYY	Y	2 * (q/3)^3

sum of these 4 probs is p(s1 s2 = 11)

dividing these 4 probs by p(11) gives $\vec{q}_{11}$

Exercise:
Check that the sum of all 16 probs = 1. Check that each of the 64 Pauli's appears exactly once in the 4 tables combined.

Detail for (5) ctd

e.g., q = 0.1894, 1-q = 0.8106, q/3 = 0.063133

Pauli's giving s1s2=00				net effect from A to B1	prob	$\vec{\xi}_{00}$
III	ZZI	IZZ	ZIZ	I	0.5423156	0.8105999
ZII	IZI	IIZ	ZZZ	Z	0.1247011	0.1863909
XXX	YYX	XYY	YXY	X	0.0010066	0.0015046
YXX	XYX	XXY	YYY	Y	0.0010066	0.0015046
					sum = 0.66904	$H(\vec{\xi}_{00}) = 0.72551$

< 1 !!

Pauli's giving s1s2=10				net effect from A to B1	prob	$\vec{\xi}_{10}$
IXI	ZYI	IYZ	ZXZ	I	0.0481966	0.436867
ZXI	IYI	IXZ	ZYZ	Z	0.0481966	0.436867
XIX	YZX	XZY	YIY	X	0.0069651	0.063133
YIX	XZX	XIY	YZY	Y	0.0069651	0.063133
					sum = 0.11032	$H(\vec{\xi}_{10}) = 1.5471$

> 1

Detail for (6) e.g., q = 0.1894

common input

+ve for $s_1 s_2 = 00$
-ve for $s_1 s_2 = 01$
10
11

$$3Q^{(3)}(\mathcal{N}_q) \geq I(R\rangle B_1B_2B_3)_\gamma = \sum_{s_1 s_2} p(s_1 s_2) \underbrace{I(R\rangle B_1)_{\gamma_{s_1 s_2}}}_{1-H(\vec{q}_{s_1 s_2})}$$

from previous page:

$I(R\rangle B_1B_2B_3)_\gamma \geq$ 1 - (0.66904 * 0.72551 + 3 * 0.11032 * 1.5471)

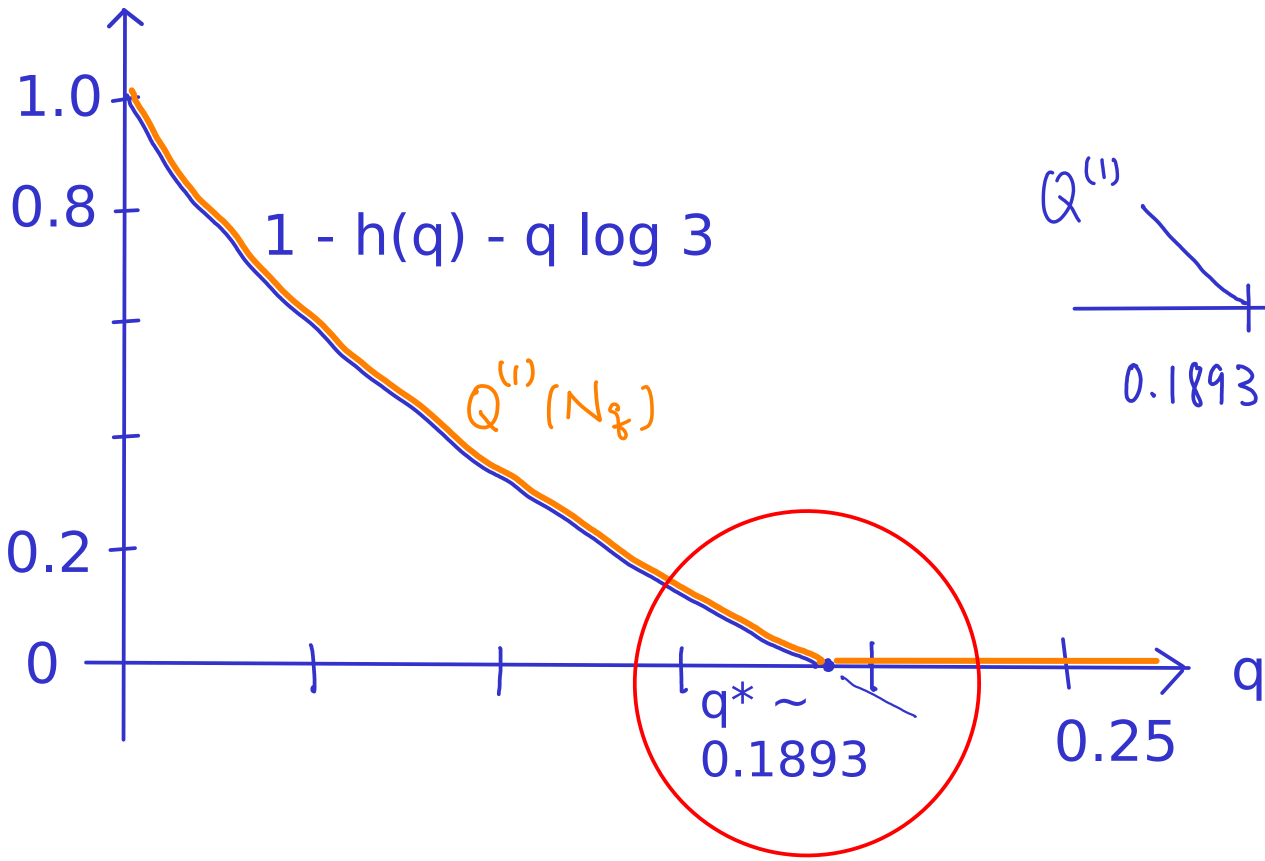
= 0.00256

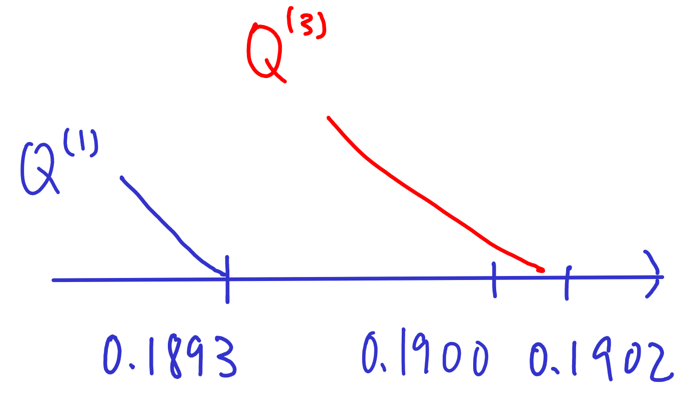
s1s2 = 01, 10, 11
have same prob and $\vec{q}_{s_1 s_2}$

Using the same input state,

$3Q^{(3)}(\mathcal{N}_q)$ > 0.00046 for q < 0.1900

$3Q^{(3)}(\mathcal{N}_q)$ > -0.0002 for q = 0.1902





NB not only the coherent info is superadditive, the r-shot can be positive while the 1-shot is 0. In fact, we do not know the range of q where the capacity of depolarizing is 0.

We also do not have an algorithm to determine if a channel has capacity or not.