CO781 / QIC 890:

Theory of Quantum Communication

Topic 1, part 3

What is communication of data?
The no-signalling principle
Optimality of superdense coding and teleportation

Cobits, duality of SD and TP,
and unitary gates as bidirectional channels

Recall: $\{|x\rangle\}$ basis for A,B.

qbit $\forall x \quad |x\rangle_A \longrightarrow |x\rangle_B |0\rangle_E$

cbit $\forall x \quad |x\rangle_A \longrightarrow |x\rangle_B |x\rangle_E$

cobit $\forall x \quad |x\rangle_A \longrightarrow |x\rangle_B |x\rangle_{A'} |0\rangle_E$

Resource inequalities:

qbit $\geqslant$ cobit $\geqslant$ cbit

cobit $\geqslant$ ebit

SD: 1 qbit + 1 ebit $\geqslant$ 2 cbits

TP: 2 cbits + 1 ebit $\geqslant$ 1 qbit

Today:

If ebits are free,

1 qbit $\underset{\text{TP}}{\overset{\text{SD}}{\rightleftharpoons}}$ 2 cbits

SD & TP invert each other

What if ebits are not free?

1 qbit + 1 ebit $\underset{\text{TP}^{\text{co}}}{\overset{\text{SD}}{\rightleftharpoons}}$ 2 cobits

SD & TP$^{\text{co}}$ invert each other

Recall: $\{|x\rangle\}$ basis for A,B.

qbit $\forall x \quad |x\rangle_A \longrightarrow |x\rangle_B |0\rangle_E$

cbit $\forall x \quad |x\rangle_A \longrightarrow |x\rangle_B |x\rangle_E$

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Resource inequalities:

qbit $\geqslant$ cobit $\geqslant$ cbit

cobit $\geqslant$ ebit

1 Today:

SD': 1 qbit + 1 ebit $\geqslant$ 2 cobits

TP: 2 cbits + 1 ebit $\geqslant$ 1 qbit + 2 rbits 2

TP^{co}: 2 cobits + 1 ebit $\geqslant$ 1 qbit + 2 ebits 3

interpret ...

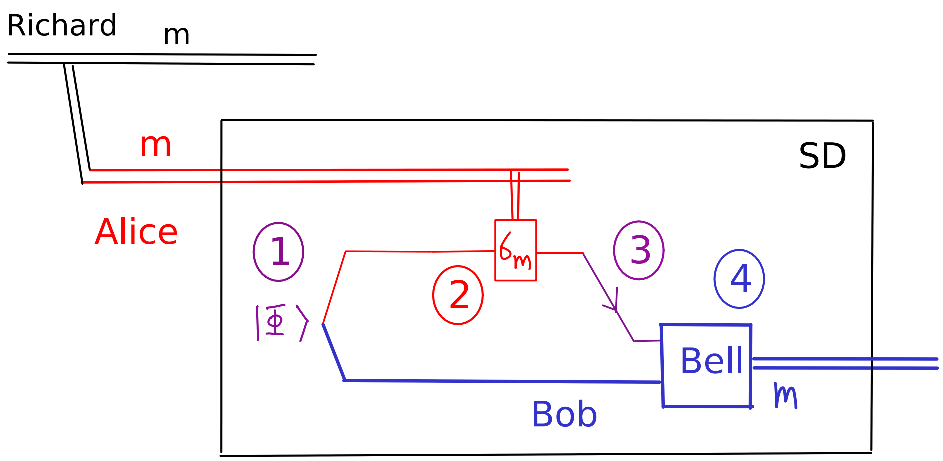
$$1 \text{ qbit} + 1 \text{ ebit} \underset{TP^{co}}{\overset{SD}{\rightleftharpoons}} 2 \text{ cobits}$$

SD & TP^{co} invert each other

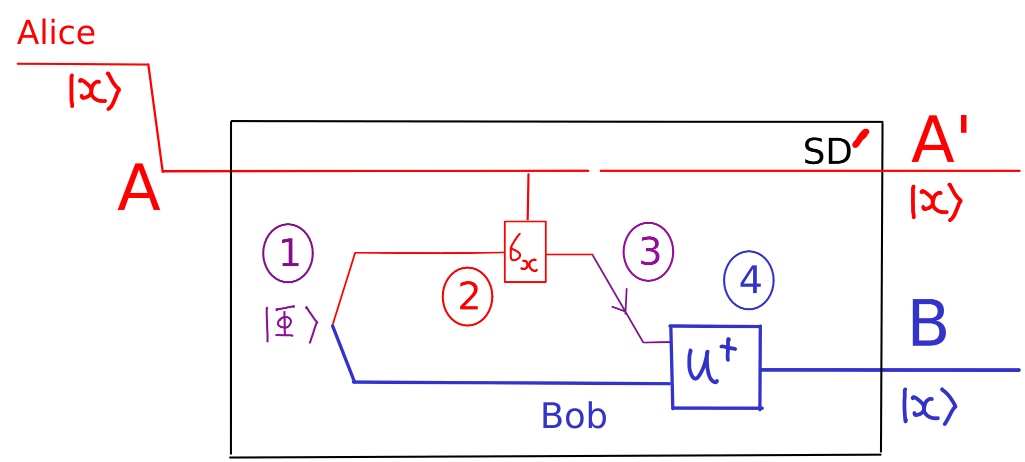
Define the basis change (for x = 0, 1, 2, 3):

$$|x\rangle \underset{U^\dagger}{\overset{U}{\rightleftarrows}} (\sigma_x \otimes I)\,|\Phi\rangle \quad , \quad |\Phi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

1



1 ebit + 1 qbit $\geq$ 2 cbits



1 ebit + 1 qbit $\geq$ 2 cobits

In short, SD leaks no info to environment.
As long as Alice & Bob keep their operations coherent,
the "classical comm" produced is coherent.

The formal state transformations:

$$|x\rangle_A \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)_{A_1B_1}$$

$\downarrow$ Alice applies controlled-σ_x, local on AA_1

$$|x\rangle_A \left[(\sigma_x \otimes I)\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)_{A_1B_1}\right]$$

$\downarrow$ use 1 qbit to comm A_1 from Alice to Bob, relabel as C

$$|x\rangle_A \left[(\sigma_x \otimes I)\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)_{CB_1}\right]$$

$\downarrow$ Bob applies $U^\dagger$ to CB_1 (local to him)

relabel output system as B

$$|x\rangle_A |x\rangle_B$$

Recall: $\{|x\rangle\}$ basis for A,B.

qbit $\forall x \quad |x\rangle_A \rightarrow |x\rangle_B |0\rangle_E$

cbit $\forall x \quad |x\rangle_A \rightarrow |x\rangle_B |x\rangle_E$

cobit $\forall x \quad |x\rangle_A \rightarrow |x\rangle_B |x\rangle_{A'} |0\rangle_E$

<u>Resource inequalities:</u>

qbit $\geq$ cobit $\geq$ cbit

cobit $\geq$ ebit

1 Today:

SD': 1 qbit + 1 ebit $\geq$ 2 cObits

TP: 2 cbits + 1 ebit $\geq$ 1 qbit + 2 rbits 2

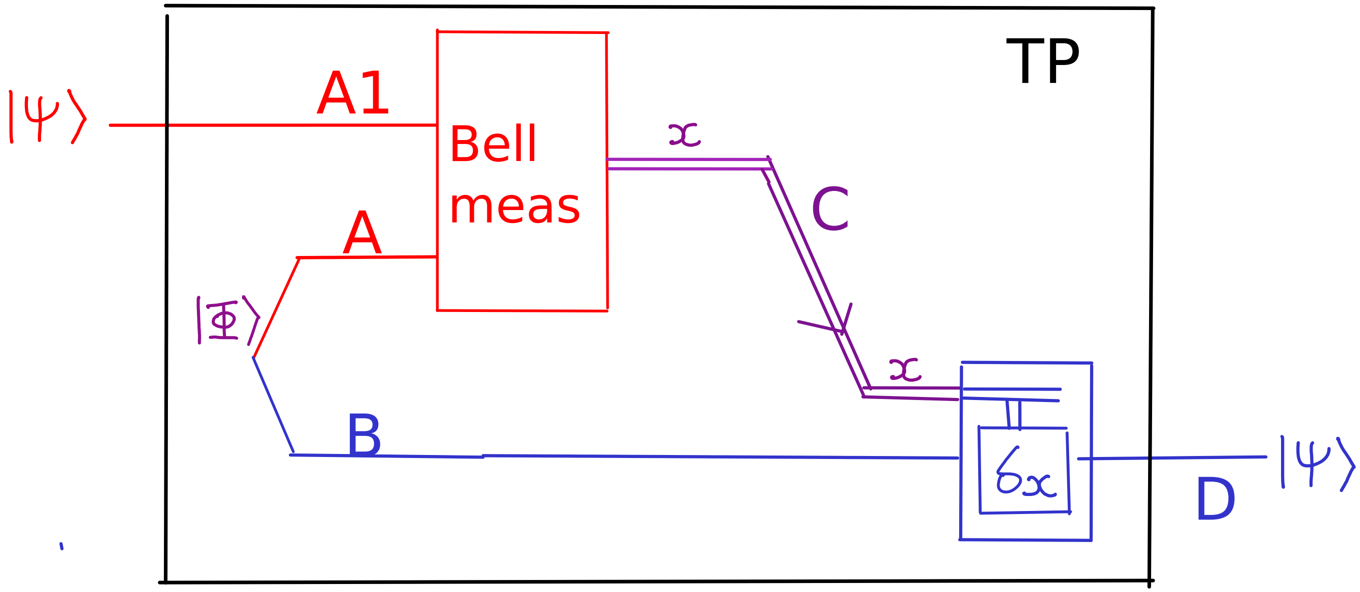
TP^{co}: 2 cobits + 1 ebit $\geq$ 1 qbit + 2 ebits 3

interpret ...

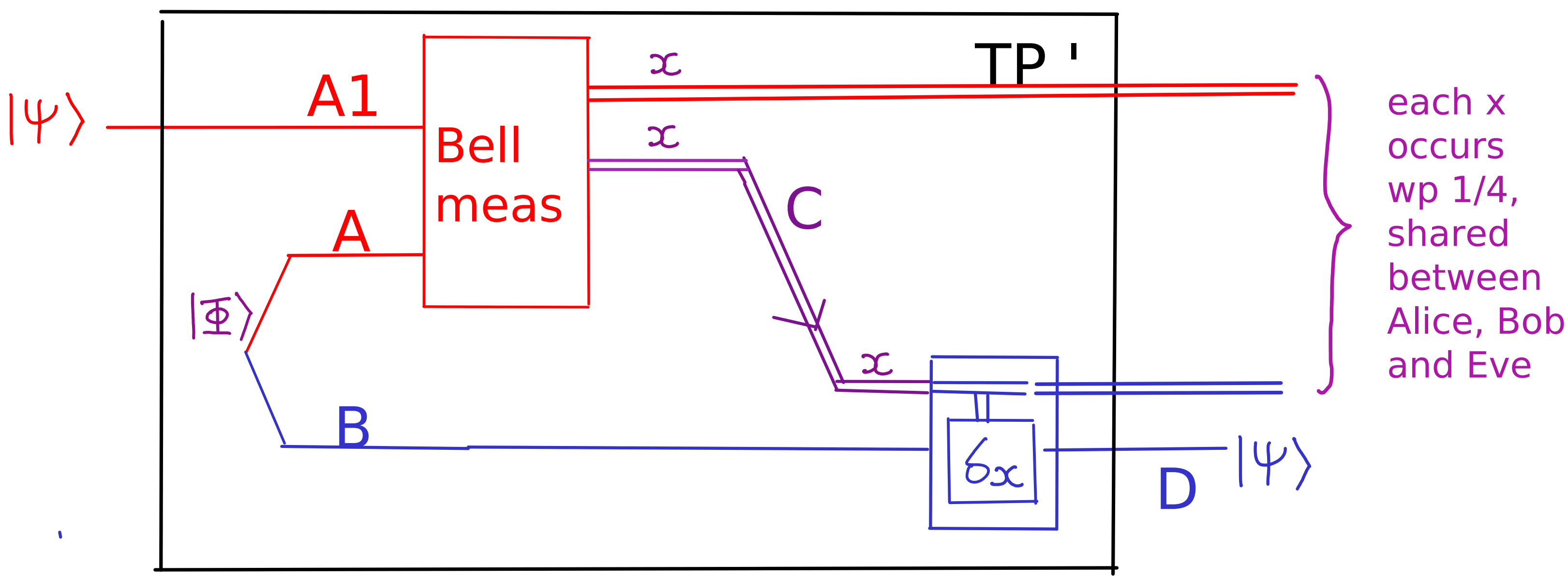
$$1 \text{ qbit} + 1 \text{ ebit} \underset{\text{TP}^{\text{co}}}{\overset{\text{SD}}{\rightleftarrows}} 2 \text{ cobits}$$

SD & TP^{co} invert each other

2



$$2 \text{ cbits} + 1 \text{ ebit} \geq 1 \text{ qbit}$$



$$2 \text{ cbits} + 1 \text{ ebit} \geq 1 \text{ qbit} + 2 \text{ rbits}$$

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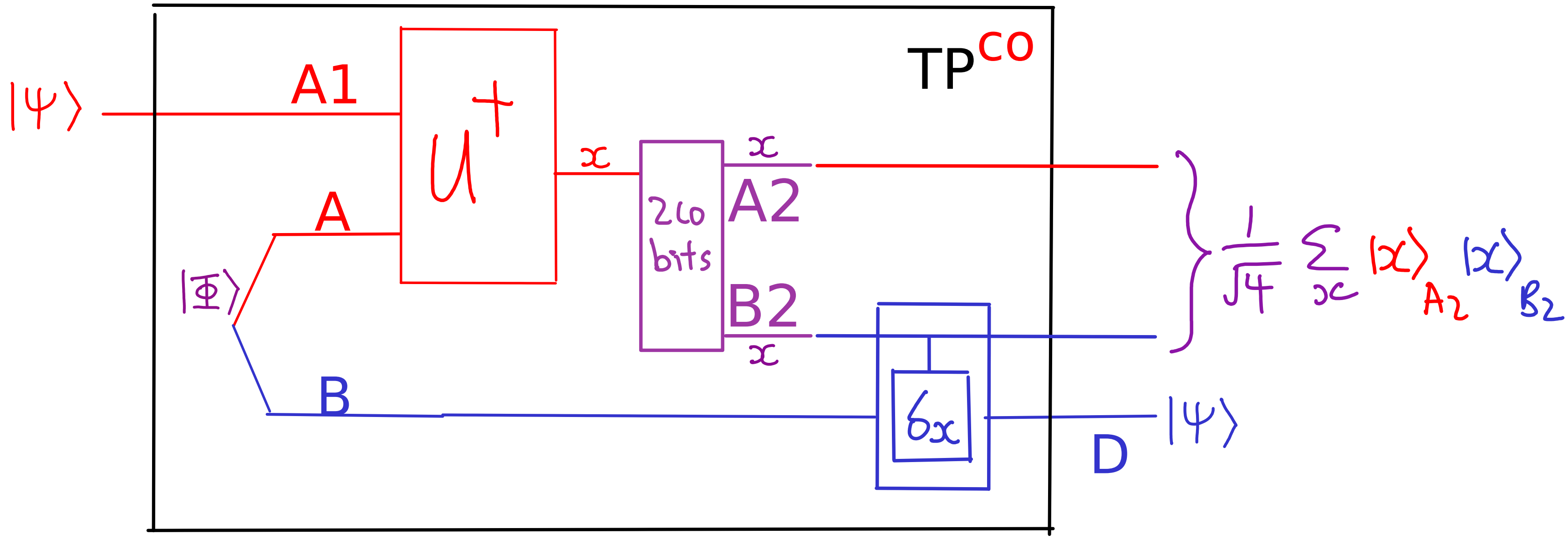
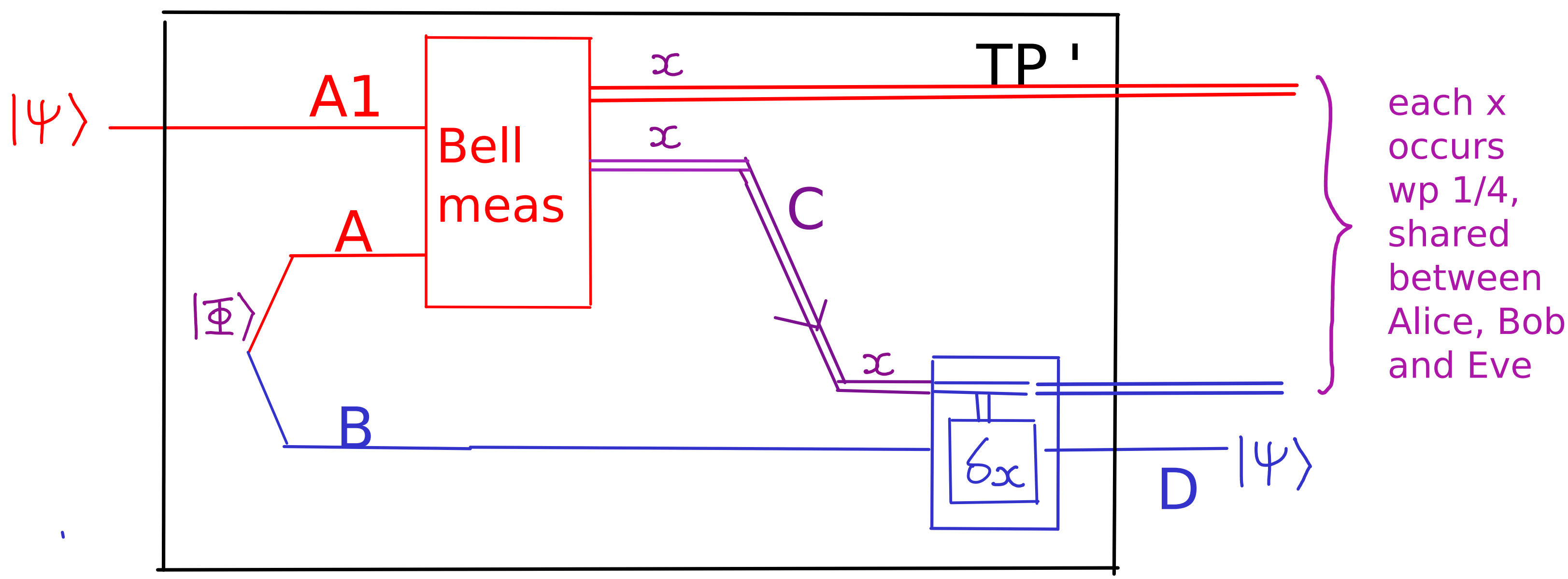


2 cObits + 1 ebit $\geqslant$ 1 qbit + 2 ebits



2 cbits + 1 ebit $\geqslant$ 1 qbit + 2 rbits

The formal state transformations:

$$|\psi\rangle_{A_1} |\Phi\rangle_{AB} = \frac{1}{2} \left[(\sigma_x \otimes I) |\Phi\rangle \right]_{A_1 A} \otimes \left[\sigma_x |\psi\rangle \right]_B \quad \text{(TP)}$$

$\downarrow$ Alice applies U^{-1} on $A_1 A$ (local),

dim(A) = dim(A_1) = 2
dim(A_3) = dim(A_2) = 4

$$= \frac{1}{2} |x\rangle_{A_3} \otimes \left[\sigma_x |\psi\rangle \right]_B$$

$\downarrow$ 2 cobits

$$= \frac{1}{2} |x\rangle_{A_2} |x\rangle_{B_2} \otimes \left[\sigma_x |\psi\rangle \right]_B$$

$\downarrow$ Bob applies controlled - $\sigma_x^{\dagger}$

local on $B_2 B$

$$= \underbrace{\frac{1}{2} |x\rangle_{A_2} |x\rangle_{B_2}}_{\text{2 ebits decoupled from everything else}} \otimes \left[|\psi\rangle \right]_B$$

2 ebits decoupled
from everything else

qbit achieved

Thus, we have the resource inequality:

$$1 \text{ ebit} + 2 \text{ cobits} \geqslant 1 \text{ qbit} + 2 \text{ ebits}$$

Repeat protocol n times sequentially, use 1 ebit of 2 from the previous round if possible:

$$1 \text{ ebit} + 2n \text{ cobits} \geqslant n \text{ qbits} + (n+1) \text{ ebits}$$

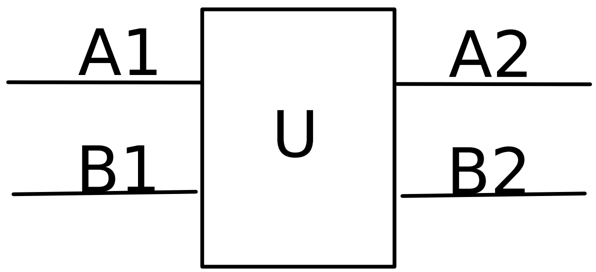
Asymptotically (for large n):

$$2 \text{ cobits} \geqslant 1 \text{ qbit} + 1 \text{ ebit}$$

which gives an interpretation of TP^{co} as "the reverse" of SD (in terms of the resource inequality).

Unitary gates as bidirectional quantum channels:

Consider a two-input, two-output unitary gate U:



Suppose A1, A2 belong to Alice, B1, B2 belong to Bob.

The gate can be used in protocols, resulting in comm of quantum or classical data in either direction or generation of entanglement.

In turns, we can use communication and entanglement to simulate these gates.

These are naturally described by resource inequalities.

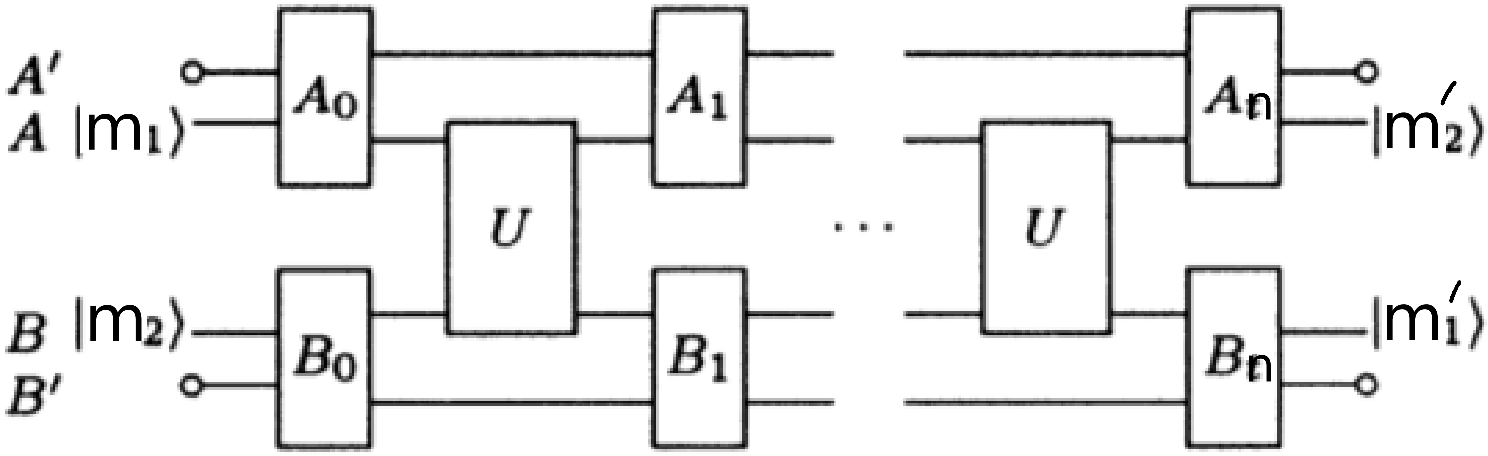
e.g., U = SWAP

SWAP $\geq$ 1 qbit $\longrightarrow$ + 1 qbit $\longleftarrow$ $\geq$ 2 ebits

Likewise: 1 qbit $\longrightarrow$ + 1 qbit $\longleftarrow$ $\geq$ SWAP

So, 1 qbit $\longrightarrow$ + 1 qbit $\longleftarrow$ = SWAP

A general communication protocol with n uses of U:



If $m_1' = m_1$, $m_2' = m_2$ with high prob, and

$m_1 \in \{1, 2, \ldots, 2^{rn}\}$, $m_2 \in \{1, 2, \ldots, 2^{sn}\}$

we say (r,s) is an achieveable rate pair.

If A'B' is in a product state, the rate pair is "unassisted".
If A'B' can be arbitrarily entangled, the rate pair is "entanglement assisted".

e.g., U = SWAP $= 1$ qbit $\rightarrow$ + 1 qbit $\leftarrow$

In the entanglement assisted case:

1. (r,s) = (2,2) achievable (by SD in both directions)

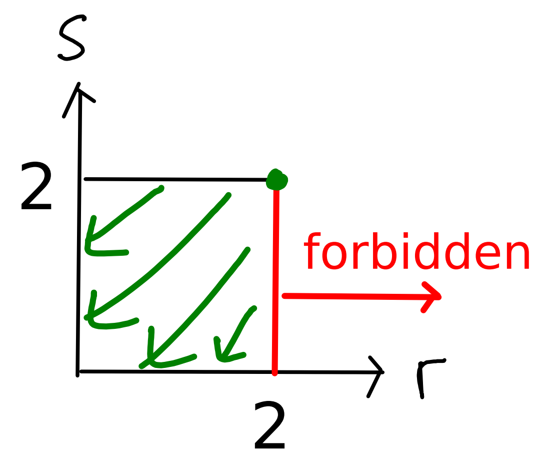
2. $r \leqslant 2$, why?

   We can replace each SWAP of n in the communication protocol using teleportation (consuming 2 ebits, 2 cbits $\rightarrow$, and 2 cbits $\leftarrow$). New protocol consumes 2n cbits from Alice to Bob, back communication, and entanglement. By C1, $rn \leqslant 2n$, so, $r \leqslant 2$.

3. Similarly, $s \leqslant 2$.

4. Known "inner bound" thus coincide with "outer bound" -- so achieveable region is known precisely.

5. No trade-off between r and s.



note monotonicity

e.g., U = SWAP $= 1$ qbit $\rightarrow$ + 1 qbit $\leftarrow$

In the unassisted case:

1. (r,s) = (2,0) achievable

For the 1st use: SWAP $\geqslant$ 1 cbit $\rightarrow$ + 1 ebit

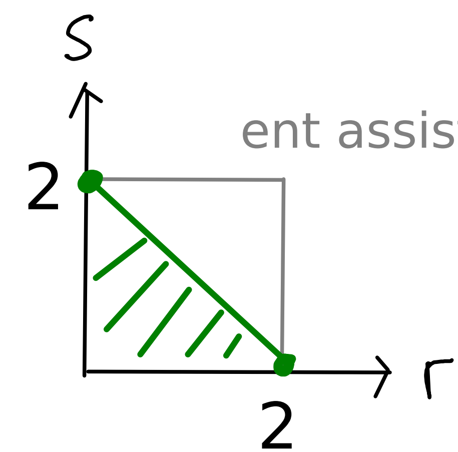
For the 2nd use: SWAP + 1 ebit $\geqslant$ 2 cbits $\rightarrow$ + 1 ebit

For the 3rd use: SWAP + 1 ebit $\geqslant$ 2 cbits $\rightarrow$ + 1 ebit

⋮

n SWAP $\geqslant$ (2n-1) cbits $\rightarrow$ (note asymptotic)

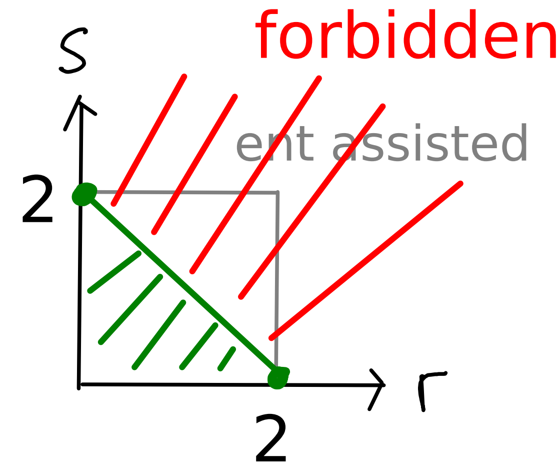
2. Similarly, (r,s) = (0,2) achievable.



3. We can show $r+s \leqslant 2$.

a. $E(\text{SWAP}) \leqslant 2$

"entanglement capacity" (# ebits created per use asymptotically)



not increased by catalytic entanglement

Proof: SWAP can be simulated with 2 ebits + classical comm. A protocol with n uses of SWAP can be turned to a protocol using 2n ebits + classical communication.

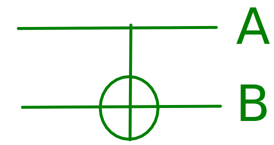
Classical comm cannot increase entanglement.

Thus $E(\text{SWAP}) \leqslant 2$.

b. $E(\text{SWAP}) \geqslant r+s$, since comm in both directions can be turned coherent, and cobit $\geqslant$ ebit

4. Exact ach rate region: triangle. Tight trade-off for r,s .

e.g., U = CNOT = $|0\rangle\langle 0|_A \otimes I_B + |1\rangle\langle 1|_A \otimes \sigma_{x\,B}$

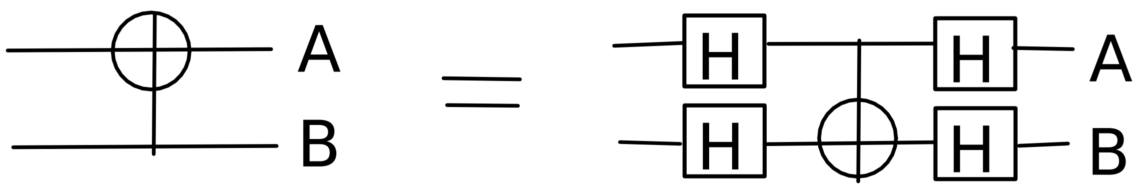


- If Bob inputs $|0\rangle_B$, Alice inputs $|x\rangle_A$ for x = 0, 1

  then CNOT $|x\rangle_A |0\rangle_B = |x\rangle_A |x\rangle_B$

  So, CNOT $\geq$ cobit $\rightarrow$

- Since



  CNOT with A, B interchanged locally equiv to original CNOT

  So, CNOT $\geq$ cobit $\leftarrow$ Ex: write down the protocol

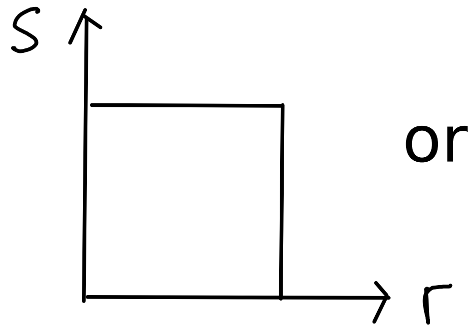
- If Bob inputs $|0\rangle_B$, Alice inputs $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_A$

  then CNOT $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_A |0\rangle_B = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB}$

  So, CNOT $\geq$ ebit

Questions to think about...

1. Is there a trade-off between forward and backward communication? Does the achievement pairs (r,s) look like:



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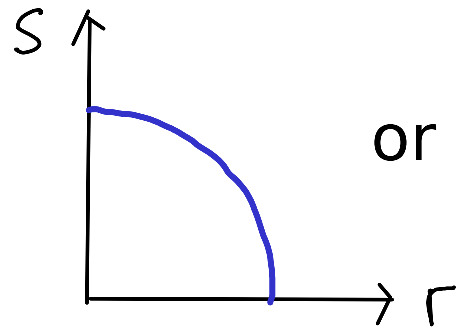

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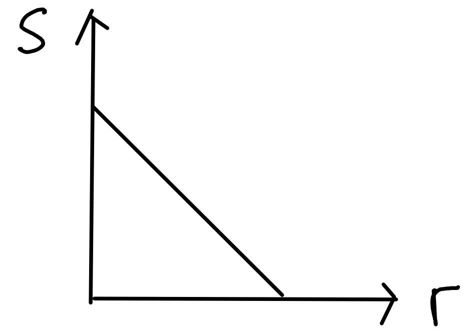


2. Does free entanglement expand the rate region?

3. What are the resources required to simulate 1 CNOT?

   Why each of 1 ebit, 1cbit $\rightarrow$, 1 cbit $\leftarrow$ are necessary?

4. Does it hold that: CNOT $\geq$ qbit $\rightarrow$?

5. Is it possible to simultaneously comm 1 bit in both directions using 1 CNOT (unassisted, or ent-assisted)?

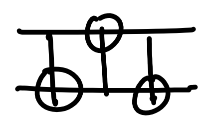
In A1, you will answer all of the above ... and magically, for each scenario (ent-assisted or unassisted), rate region for CNOT is exactly half of rate region for SWAP !!

You will see that CNOT + ebit = cobit $\rightarrow$ + cobit $\leftarrow$

So 2 CNOT + 2 ebits = 2 cobits $\rightarrow$ + 2 cobits $\leftarrow$

= 1 qbit $\rightarrow$ + 1 ebit + 1 qbit $\leftarrow$ + 1 ebit

= SWAP + 2 ebits

So, 2 CNOT = SWAP !! (asymptotic) (cf SWAP = )

NB cobit relates TP w/ SD, CNOT w/ SWAP etc and the "family of protocols" due to Devetak-Harrow-Winter.

Remarks on bidirectional gates:

1. A gate is either product (all rates 0) or entangled (all rates are positive).

2. generally, rates are not symmetric wrt interchanging Alice and Bob.

3. entanglement capacity can be >> rate sum.

4. can consider Hamiltonians instead of gates

5. see quant-ph/0205057, quant-ph/0307091

Lots of possibilities for term projects,