Bernoulli Trial 2024

1: (3 minutes) **T/F**: There is a complex number z with $|z| = 1$ such that

$$z^{2024} + z^4 + z^2 + 1 = 0.$$

2: (3 minutes) Jerry is making a MATH 145 final exam which consists of 40 T/F questions by the following inductive algorithm. Assume that questions $1, 2, \ldots, n-1$ have been chosen.

- If there are more T than F among them, then question n will be F with probability 0.69.
- If there are more F than T among them, then question n will be T with probability 0.69.
- If there are equal number of T and F among them, then question n will be T with probability 0.5.

T/F: The expected number of T questions is 20.

3: (3 minutes) **T/F**: There exist positive rational numbers $a_1, a_2, \ldots, a_{69}$ (not necessarily distinct) such that

$$\sum_{i=1}^{69} a_i = \prod_{i=1}^{69} a_i = 420.$$

4: (3 minutes) **T/F**: There exists a 2024-digit positive integer with only 6 or 9 appearing, that is divisible by 2^{2024}.

5: (3 minutes) **T/F**: For any prime number $p \geq 3$, the integral

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x-1}{x^p - 1} dx$$

is an algebraic number, that is the root of some nonzero polynomial with rational coefficients.

6: (3 minutes) **T/F**: There does not exist a continuous function $f : [0,1] \to \mathbb{R}$ such that each pre-image $f^{-1}(b) = \{a \in [0,1] \colon f(a) = b\}$ is a (possibly empty) finite set of even size.

7: (3 minutes) **T/F**: The smallest positive integer d such that $2027^d \equiv 1 \pmod{49}$ is 3.

8: (2 minutes) Let $a_1, a_2, \ldots, a_{880}$ denote all the integers $1, 2, \ldots, 2024$ that are coprime to 2024.

T/F:

$$\sum_{k=1}^{880} a_k^{2024} \equiv 880 \pmod{2024}.$$

9: (3 minutes) **T/F**: There are nonzero integers a, b, c such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 0.$$

10: (4 minutes) A cyclic number is a positive integer for which cyclic permutations of the digits are successive integer multiples of the number. (Leading zeros are allowed.) For example, 142857 is cyclic:

$$\begin{aligned} 142857 \times 1 &= 142857 \\ 142857 \times 2 &= 285714 \\ 142857 \times 3 &= 428571 \\ 142857 \times 4 &= 571428 \\ 142857 \times 5 &= 714285 \\ 142857 \times 6 &= 857142. \end{aligned}$$

T/F: If p is a Fermat prime at least 17, i.e. a prime number of the form $2^{2^n} + 1$, then $\dfrac{10^{p-1} - 1}{p}$ is a cyclic number.

11: (4 minutes) **T/F**: The Euclidean space $\mathbb{R}^3$ can not be covered by pairwise non-coplanar lines.

12: (3 minutes) **T/F**:

$$\int_0^\infty \frac{dx}{(1+x^2)(1+x^{2024})} < \frac{\pi}{4}.$$

13: (4 minutes) **T/F**: Given any sequence $a_1, a_2, \ldots, a_{2024}$ of 2024 distinct real numbers, either there is an increasing subsequence of length 120 or a decreasing subsequence of length 18.

14: (4 minutes) Gian is participating in the hardcore Bernoulli trial where answering a question correctly gives 1 point and incorrectly loses 1 point. Gian starts with 1 point and will be eliminated when he has only 0 points. When Gian has n points, he will answer the next question correctly with probability $\frac{(n+1)^2}{n^2+(n+1)^2}$.

For example, with 1 point, Gian answers the next question correctly with probability 4/5; with 2 points, the probability is 9/13; with a lot of points, Gian is basically guessing.

T/F: The probability that Gian will play forever is at most 60%.

15: (5 minutes) **T/F**:

$$15! = 1307674368000.$$

16: Tie break, if needed (3 minutes) Compute $\pi^4 + \pi^5 - e^6$.