Bernoulli Trial 2024

1. There are 15 questions. Answer T or F only.
2. Put your name, your BT ID, and the question number on each answer slip.
3. Correct answer = 1 point; Incorrect answer = -1 point; No answer = 0 point.
4. At least 12 questions need to be answered to qualify for the prizes.
5. Prizes:
   - First place: $200
   - Second place: $100
   - Third place: $50
   - Last place: $200

1: (3 minutes)

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$$z^{2024} + z^4 + z^2 + 1 = 0.$$

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F: Note that if $z, w \in \mathbb{C}$ with $|z| = |w| = 1$, then $|z + w|^2 = 2 + 2\cos\theta(z, w)$ where $\theta(z, w)$ is the angle between z and w. Hence, in order for $z^{2024} + z^4 = -z^2 - 1$, the angle between 1 and z^2 is the same as the angle between z^4 and z^{2024}. Write $z = e^{i\theta}$. Then $2\theta = \pm 2020\theta + 2k\pi$ for some integer k. So either $z^{2022} = 1$ or $z^{2018} = 1$.

If $z^{2022} = 1$, then we have $(z^2+1)^2 = 0$ and so $z^2 = -1$ and $z^{2022} = -1$. If $z^{2018} = 1$, then $z^6+z^4+z^2+1 = 0$ which implies $z^8 = 1$, and so $z^2 = 1$ and it is also impossible.

Gian's Lemma: Given four complex numbers on the unit circle that sum to 0, they must form a rectangle.

2: (3 minutes)

Jerry is making a MATH 145 final exam which consists of 40 T/F questions by the following inductive algorithm. Assume that questions $1, 2, \ldots, n-1$ have been chosen.

- If there are more T than F among them, then question n will be F with probability 0.69.
- If there are more F than T among them, then question n will be T with probability 0.69.
- If there are equal number of T and F among them, then question n will be T with probability 0.5.

T/F: The expected number of T questions is 20.

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- If there are more F than T among them, then question n will be T with probability 0.69.
- If there are equal number of T and F among them, then question n will be T with probability 0.5.

T/F: The expected number of T questions is 20.

T: Let X_n be the number of T questions minus the number of F questions among questions $1, 2, \ldots, n$. For example $P(X_1 = 1) = P(X_1 = -1) = 0.5$. By symmetry, one can prove by induction on n that for any positive integer m, we have

$$P(X_n = m) = P(X_n = -m).$$

Hence $E(X_n) = 0$ for all n.

3: (3 minutes)

T/F: There exist positive rational numbers $a_1, a_2, \ldots, a_{69}$ (not necessarily distinct) such that

$$\sum_{i=1}^{69} a_i = \prod_{i=1}^{69} a_i = 420.$$

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T: Using $a_1 = 3$, $a_2 = 4$, $a_3 = 5$, $a_4 = 7$, we see that it remains to find 65 rational numbers that multiply to 1 and sum to 401. The 4 rational numbers $1/3, 1/3, 1/3, 27$ multiply to 1 and sum to 28. By taking 14 copies of them, we reduce to finding 9 rational numbers that multiply to 1 and sum to $401 - 28 \cdot 14 = 9$. We simply take all 1 now.

4: (3 minutes)

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T: We prove by induction that there is always an n-digit integer of that form that is divisible by 2^n. The key is simply that

$$2^{n+1} \,||\, 6 \cdot 10^n, \qquad 2^n \,||\, 9 \cdot 10^n.$$

So if the n-digit number is divisible by 2^{n+1}, we add $6 \cdot 10^n$ to it. If it is not divisible by 2^{n+1}, we add $9 \cdot 10^n$ to it.

5: (3 minutes)

T/F: For any prime number $p \geq 3$, the integral

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x-1}{x^p-1}dx$$

is an algebraic number, that is the root of some nonzero polynomial with rational coefficients.

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T: Let

$$f(x) = \frac{x^p - 1}{x - 1} = \prod_{k=1}^{p-1}(x - \zeta_p^k).$$

For a large R, let C_R be the upper semicircle of radius R centered at the origin, traversed counterclockwise. Then

$$\int_{-R}^{R} \frac{1}{f(x)}dx + \int_{C_R} \frac{1}{f(z)}dz = 2\pi i \sum_{k=1}^{(p-1)/2} \operatorname{res}(\frac{1}{f}; \zeta_p^k) = 2\pi i \sum_{k=1}^{(p-1)/2} \frac{1}{f'(\zeta_p^k)}.$$

On C_R, we have $|z| = R$ and $|f(z)| \geq (R-1)^{p-1}$. Hence, as $R \to \infty$, we have

$$\int_{C_R} \frac{1}{f(z)}dz \leq \frac{2\pi R}{(R-1)^{p-1}} \to 0.$$

Therefore,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x-1}{x^p-1}dx = 2i\sum_{k=1}^{(p-1)/2}\frac{1}{f'(\zeta_p^k)} \in \bar{\mathbb{Q}}.$$

With a bit of calculation, one has

$$\sum_{k=1}^{(p-1)/2}\frac{1}{f'(\zeta_p^k)} = \frac{1}{p}\sum_{k=1}^{(p-1)/2}\zeta_p^{2k}-\zeta_p^k.$$

6: (3 minutes)

T/F: There does not exist a continuous function $f : [0,1] \to \mathbb{R}$ such that each pre-image

$$f^{-1}(b) = \{a \in [0,1] \colon f(a) = b\}$$

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F: Draw a zigzag picture.

7: (3 minutes)

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T: Note that

$$49 \times 41 = 2025 - 16 = 2009.$$

So $2027 \equiv 18 \pmod{49}$. Now by Gian's identity

$$19^2 - 19 + 1 = 343,$$

we know that

$$18 \equiv 19^2 \pmod{343}$$

and

$$18^3 \equiv 19^6 \equiv 1 \pmod{343}.$$

The last congruence uses

$$x^6 - 1 = (x-1)(x+1)(x^2 - x + 1)(x^2 + x + 1).$$

8: (2 minutes)

Let $a_1, a_2, \ldots, a_{880}$ denote all the integers $1, 2, \ldots, 2024$ that are coprime to 2024.

T/F:

$$\sum_{k=1}^{880} a_k^{2024} \equiv 880 \pmod{2024}.$$

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T: We factor $2024 = 8 \cdot 11 \cdot 23$. Note that $22 \mid 2024$ and $2 \mid 2024$. So we have

$$\begin{aligned} a_k^{2024} &\equiv 1 \pmod{23} \\ a_k^{2024} &\equiv 1 \pmod{8} \\ a_k^{2024} &\equiv a_k^4 \pmod{11} \end{aligned}$$

We know

$$\sum_{a \in (\mathbb{Z}/11\mathbb{Z})^\times} a^4 = 0 \in \mathbb{Z}/11\mathbb{Z}.$$

Hence the desired sum is 880 mod 23, 880 mod 8, and 0 mod 11. By CRT, it is 880 mod 2024.

9: (3 minutes)

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$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 0.$$

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F: Gian said it follows from some ν_p (p-adic valuation) analysis. I killed it with a hammer. Let $u = a/b$, $v = b/c$, $w = c/a$. Then the cubic

$$f(x) = (x-u)(x-v)(x-w) = x^3 - Ax - 1$$

for some $A \in \mathbb{Q}$. In order for all the roots of $f(x)$ to be in $\mathbb{Q}$, its discriminant

$$\Delta(f) = (u-v)^2(v-w)^2(w-u)^2 = 4A^3 - 27$$

is a square in $\mathbb{Q}$. However, the elliptic curve $y^2 = 4x^3 - 27$ has only $(3, \pm 9)$ as its rational points. It is then easy to see that $x^3 - 3x - 1$ doesn't have rational roots.

To find the rational points on $y^2 = 4x^3 - 27$, we note that

$$(9+y)^3 + (9-y)^3 = 2 \cdot 9^3 + 2 \cdot 3 \cdot 9y^2 = (6x)^3.$$

By Fermat's Last Theorem, we have $y = \pm 9$.

10: (4 minutes)

A cyclic number is a positive integer for which cyclic permutations of the digits are successive integer multiples of the number. (Leading zeros are allowed.) For example, 142857 is cyclic:

$$\begin{aligned}
142857 \times 1 &= 142857 \\
142857 \times 2 &= 285714 \\
142857 \times 3 &= 428571 \\
142857 \times 4 &= 571428 \\
142857 \times 5 &= 714285 \\
142857 \times 6 &= 857142.
\end{aligned}$$

T/F: If p is a Fermat prime at least 17, i.e. a prime number of the form $2^{2^n}+1$, then $\dfrac{10^{p-1}-1}{p}$ is a cyclic number.

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A cyclic number is a positive integer for which cyclic permutations of the digits are successive integer multiples of the number. (Leading zeros are allowed.) For example, 142857 is cyclic.

T/F: If p is a Fermat prime at least 17, i.e. a prime number of the form $2^{2^n}+1$ with $n \geq 2$, then $\dfrac{10^{p-1}-1}{p}$ is a cyclic number.

T: It is known that $\frac{10^{p-1}-1}{p}$ is cyclic if and only if 10 is primitive mod p, in the sense that $o_p(10) = p-1$. Since $p-1$ is a power of 2, this is equivalent to $10^{(p-1)/2} \not\equiv 1 \pmod{p}$. We now compute Legendre symbols. Since $p \equiv 1 \pmod{8}$, we have $\left(\frac{2}{p}\right) = 1$. Moreover, for $n \geq 2$, we have $4 \mid 2^n$ and so $2^{2^n} \equiv 1 \pmod{5}$. By quadratic reciprocity

$$\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right) = \left(\frac{2}{5}\right) = -1.$$

Hence $\left(\frac{10}{p}\right) = -1$.

11: (4 minutes)

T/F: The Euclidean space $\mathbb{R}^3$ can not be covered by pairwise non-coplanar lines.

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F: Let's just try to do it. Every line will pass through the $z = 0$ plane at some $(a, b, 0)$, with directional vector $(u(a, b), v(a, b), 1)$ and every $(a, b) \in \mathbb{R}^2$ is needed. For two such lines to intersect at a point with z-coordinate t, we need

$$\begin{aligned} a + tu(a, b) &= a' + tu(a', b') \\ a + tv(a, b) &= a' + tv(a', b'). \end{aligned}$$

So

$$\frac{a - a'}{u(a, b) - u(a', b')} = \frac{b - b'}{v(a, b) - v(a', b')}.$$

By taking $u(a, b) = b$ and $v(a, b) = -a$, we see that this becomes

$$(a - a')^2 + (b - b')^2 = 0.$$

That is, the distinct lines will never intersect, and are also not parallel. To find the line passing through any (x, y, z), we see that

$$\begin{aligned} a + zb &= x \\ b - za &= y \end{aligned}$$

always has a solution for a, b since the determinant is $1 + z^2 \neq 0$.

12: (3 minutes)

T/F:

$$\int_0^\infty \frac{dx}{(1+x^2)(1+x^{2024})} < \frac{\pi}{4}.$$

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$$\int_0^\infty \frac{dx}{(1+x^2)(1+x^{2024})} < \frac{\pi}{4}.$$

T: They are equal. In fact, for any $r > 0$,

$$\begin{aligned}
\int_0^\infty \frac{dx}{(1+x^2)(1+x^r)} &= \int_0^1 \frac{dx}{(1+x^2)(1+x^r)} + \int_1^\infty \frac{dx}{(1+x^2)(1+x^r)} \\
&= \int_1^\infty \frac{x^r\,dx}{(1+x^2)(1+x^r)} + \int_1^\infty \frac{dx}{(1+x^2)(1+x^r)} \\
&= \int_1^\infty \frac{dx}{(1+x^2)} \\
&= \frac{\pi}{4}.
\end{aligned}$$

13: (4 minutes)

T/F: Given any sequence $a_1, a_2, \ldots, a_{2024}$ of 2024 distinct real numbers, either there is an increasing subsequence of length 120 or a decreasing subsequence of length 18.

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T: For every $i = 1, 2, \dots, 2024$, define a pair $(r_i, s_i) \in \mathbb{N}^2$ where r_i is the length of the longest increasing subsequence ending with a_i, and s_i is the length of the longest decreasing subsequence starting with a_i. Suppose $i < j$. If $a_i < a_j$, then by adding a_j to a maximal increasing subsequence ending with a_i, we have $r_j > r_i$. Similarly, if $a_i > a_j$, then $s_i > s_j$. Hence, the pairs (r_i, s_i) are distinct as $i = 1, \dots, 2024$. The number of distinct pairs (r, s) with $r = 1, \dots, 119$ and $s = 1, \dots, 17$ is $119 \cdot 17 = 2023 < 2024$. We are done by Pigeonhole.

14: (4 minutes)

Gian is participating in the hardcore Bernoulli trial where answering a question correctly gives 1 point and incorrectly loses 1 point. Gian starts with 1 point and will be eliminated when he has only 0 points. When Gian has n points, he will answer the next question correctly with probability $\frac{(n+1)^2}{n^2+(n+1)^2}$.

For example, with 1 point, Gian answers the next question correctly with probability 4/5; with 2 points, the probability is 9/13; with a lot of points, Gian is basically guessing.

T/F: The probability that Gian will play forever is at most 60%.

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T/F: The probability that Gian will play forever is at most 60%.

F: The actual probability is $6/\pi^2 > 0.6$. Let p_n be the probability that 0 point is reached with a starting point of n, with $p_0 = 1$. Then

$$p_n = \frac{(n+1)^2}{n^2+(n+1)^2} p_{n+1} + \frac{n^2}{n^2+(n+1)^2} p_{n-1}.$$

So

$$p_{n+1} - p_n = \frac{n^2}{(n+1)^2}(p_n - p_{n-1}) = \cdots = \frac{1}{(n+1)^2}(p_1 - p_0).$$

Summing gives

$$p_{n+1} - 1 = \sum_{k=1}^{n+1} \frac{1}{k^2}(p_1 - 1).$$

Consider the functional equation $f(0) = 1$ and

$$f(n) = \sum_{j=0}^{\infty} p_{n,j} f(j)$$

where

$$p_{n,n+1} = \frac{(n+1)^2}{n^2 + (n+1)^2}, \qquad \text{and} \qquad p_{n,n-1} = \frac{n^2}{n^2 + (n+1)^2}, \qquad \text{and} \qquad p_{n,j} = 0 \text{ otherwise.}$$

Then we see that $f(n) = p_n$ is a solution. The key here is that it is the smallest non-negative solution! Suppose f is any solution. Let X_k be Gian's score after question k assuming that Gian starts with n points. Then $p_{n,0} = P(X_1 = 0)$ and so

$$f(n) = P(X_1 = 0) + \sum_{j=1}^{\infty} p_{n,j} f(j) = P(X_1 = 0) + \sum_{j=1}^{\infty} p_{n,j}(p_{j,0} + \sum_{\ell=1}^{\infty} p_{j,\ell} f(\ell)).$$

Note that

$$\sum_{j=1}^{\infty} p_{n,j} p_{j,0} = P(X_1 \neq 0, X_2 = 0).$$

Since f is always non-negative, we have

$$f(n) \geq P(X_1 = 0) + P(X_1 \neq 0, X_2 = 0) + \cdots + P(X_1 \neq 0, \ldots X_m \neq 0, X_{m+1} = 0)$$

for any $m \geq 1$. Letting m go to infinity gives $f(n) \geq p_n$.

Now

$$f(n+1) = 1 - \sum_{k=1}^{n+1} \frac{1}{k^2}(1 - f(1))$$

is always a solution by the same calculation as above. If $f(1) < 1 - 6/\pi^2$, then for n large enough,

$$\sum_{k=1}^{n+1} \frac{1}{k^2}(1 - f(1)) > 1.$$

If $f(1) > 1 - 6/\pi^2$, then picking any $g(1)$ between $1 - 6/\pi^2$ and $f(1)$ gives a non-negative solution with

$$g(n+1) = 1 - \sum_{k=1}^{n+1} \frac{1}{k^2}(1 - g(1)) < 1 - \sum_{k=1}^{n+1} \frac{1}{k^2}(1 - f(1)) = f(n+1).$$

Therefore, the smallest non-negative solution happens with $p_1 = 1 - 6/\pi^2$.

15: (5 minutes)

T/F:

$$15! = 1307674368000.$$

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T: Embrace the bash.

$$\begin{aligned}
1307674368000 &= 100590336000 \times 13 \\
&= 4790016000 \times 21 \times 13 \\
&= 6842880 \times 10 \times 70 \times 21 \times 13 \\
&= 622080 \times 11 \times 10 \times 14 \times 5 \times 7 \times 3 \times 13 \\
&= 20736 \times 2 \times 15 \times 11 \times 10 \times 14 \times 5 \times 7 \times 3 \times 13.
\end{aligned}$$

We are now missing $4, 6, 8, 9, 12$. We have

$$20736 = 2304 \times 9 = 256 \times 9^2 = 2^8 \times 3^4 = 4 \times 6 \times 8 \times 9 \times 12.$$

16: Pizza/Tie break, if needed (3 minutes)

Compute

$$\frac{\ln(14112^2 + 104)}{\sqrt{37}}.$$

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Compute

$$\frac{\ln(14112^2 + 104)}{\sqrt{37}}.$$

Input

$$\frac{\log(14\,112^2 + 104)}{\sqrt{37}}$$

Result

$$\frac{\log(199\,148\,648)}{\sqrt{37}}$$

Decimal approximation

3.1415926535898113612301281440220816877290504920430955285897728550
...

This is related to the fact that

$$e^{\pi\sqrt{37}} + 24 \approx (12 + 2\sqrt{37})^6 \approx (12 + 2\sqrt{37})^6 + (12 - 2\sqrt{37})^6 \in \mathbb{Z}.$$