Solutions to the Special K Problems, 2010

1: Find the minimum possible discriminant $\Delta = b^2 - 4ac$ of a quadratic $f(x) = ax^2 + bx + c$ which satisfies the requirement that $f\big(f\big(f(0)\big)\big) = f(0)$.

Solution: Let $f(x) = ax^2 + bx + c$. Then $f(0) = c$ so we have

$$\begin{aligned} f\big(f\big(f(0)\big)\big) = f(0) &\iff f\big(f(c)\big) = c \\ &\iff f(ac^2 + bc + c) = c \\ &\iff a(ac^2 + bc + c)^2 + b(ac^2 + bc + c) + c = c \\ &\iff (ac^2 + bc + c)\big(a(ac^2 + bc + c) + b\big) = 0 \\ &\iff (ac^2 + bc + c)(a^2c^2 + abc + ac + b) = 0 \\ &\iff c(ac + b + 1)(ac + 1)(ac + b) = 0 \\ &\iff c = 0\ ,\quad ac = -(b+1)\ ,\quad ac = -1\ ,\quad \text{or } ac = -b\,. \end{aligned}$$

When $c = 0$ we have $\Delta = b^2 - 4ac = b^2 \geq 0$, when $ac = -(b+1)$ we have $\Delta = b^2 + 4(b+1) = (b+2)^2 \geq 0$, when $ac = -1$ we have $\Delta = b^2 + 4 \geq 4$, and when $ac = -b$ we have $\Delta = b^2 + 4b = (b+2)^2 - 4 \geq -4$. Thus the minimum possible value for Δ is $\Delta = -4$, and this minimum value is attained when $b = -2$ and $ac = 2$, for example when $f(x) = x^2 - 2x + 2$.

2: Show that for every integer a, there exist infinitely many perfect powers of the form

$$a + 2010\,t\ ,\ \ t \in \mathbf{Z}\,.$$

(A *perfect power* is an integer of the form n^k for some integers $n \geq 0$ and $k \geq 2$).

Solution: Note that $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, which is a product of distinct primes. We claim, more generally, that if $m = p_1 p_2 \cdots p_l$ where the p_i are distinct primes, then for every $a \in \mathbf{Z}$ there exist infinitely many perfect powers of the form $a + mt$, $t \in \mathbf{Z}$. Let $\psi \geq 1$ be any common multiple of the numbers $p_i - 1$ for which $p_i \big| a$ (for example we could take $\psi = \phi(m)$). For those values of i for which $p_i \big| a$ we have $a \equiv 0 \pmod{p_i}$ and so $a^{\psi+1} \equiv 0 \equiv a \pmod{p_i}$. For those values of i for which $p_i \not| a$, by Fermat's Little Theorem we have $a^\psi \equiv 1 \pmod{p_i}$ and so again we have $a^{\psi+1} \equiv a \pmod{p_i}$. Thus $a^{\psi+1} \equiv a \pmod{p_i}$ for all $i = 1, 2, \cdots, l$, and so by the Chinese Remainder Theorem $a^{\psi+1} \equiv a \pmod{m}$. Finally note that for any $b \geq 0$ with $b \equiv a \pmod{m}$ we have $b^{\psi+1} \equiv a^{\psi+1} \equiv a \pmod{m}$, so we have found infinitely many perfect powers $b^{\psi+1}$ of the form $a + mt$, $t \in \mathbf{Z}$.

We remark that the above argument does not work when m has a factor of the form p^2 with p prime, and indeed when $p^2 \big| m$ there are no perfect powers of the form $p + mt$, $t \in \mathbf{Z}$.

3: Let n be a positive integer. Evaluate $\displaystyle\sum_{k=0}^{\infty}\left\lfloor\frac{n+2^k}{2^{k+1}}\right\rfloor$, where $\lfloor x\rfloor$ is the largest integer less than or equal to x.

Solution: Let $S_n=\displaystyle\sum_{k=0}^{\infty}\left\lfloor\frac{n+2^k}{2^{k+1}}\right\rfloor$. We claim that $S_n=n$ for all positive integers n. Note first that

$$S_1=\lfloor\tfrac{1+1}{2}\rfloor+\lfloor\tfrac{1+2}{4}\rfloor+\lfloor\tfrac{1+4}{8}\rfloor+\cdots=1+0+0+\cdots=1\,.$$

Let $n\geq 2$ and suppose, inductively, that $S_l=l$ for all l with $1\leq l<n$. If n is even, say $n=2l$, then

$$\begin{aligned}S_n&=\lfloor\tfrac{2l+1}{2}\rfloor+\lfloor\tfrac{2l+2}{4}\rfloor+\lfloor\tfrac{2l+4}{8}\rfloor+\lfloor\tfrac{2l+8}{16}\rfloor+\cdots\\&=l+\lfloor\tfrac{l+1}{2}\rfloor+\lfloor\tfrac{l+2}{4}\rfloor+\lfloor\tfrac{l+4}{8}\rfloor+\cdots\\&=l+S_l=2l=n.\end{aligned}$$

Before considering the case that n is odd, we first claim that when a and b are both even we have $\lfloor\frac{a+1}{b}\rfloor=\lfloor\frac{a}{b}\rfloor$. To see this, let a and b be even and use the Division Algorithm to write $a=qb+r$ with $0\leq r<b$. Note that r must be odd, so $r\neq 0$, and so we have $a=qb+s$ where $s=r-1$ with $0\leq s<b$. Thus $\lfloor\frac{a+1}{b}\rfloor=q=\lfloor\frac{a}{b}\rfloor$. Now we consider the case that n is odd, say $n=2l+1$, then

$$\begin{aligned}S_n&=\lfloor\tfrac{2l+2}{2}\rfloor+\lfloor\tfrac{2l+1+2}{4}\rfloor+\lfloor\tfrac{2l+1+4}{8}\rfloor+\lfloor\tfrac{2l+1+8}{16}\rfloor+\cdots\\&=l+1+\lfloor\tfrac{2l+2}{4}\rfloor+\lfloor\tfrac{2l+4}{8}\rfloor+\lfloor\tfrac{2l+8}{16}\rfloor\cdots\\&=l+1+\lfloor\tfrac{l+1}{2}\rfloor+\lfloor\tfrac{l+2}{4}\rfloor+\lfloor\tfrac{l+4}{8}\rfloor+\cdots\\&=l+1+S_l=2l+1=n\,.\end{aligned}$$

4: A point $p = (x, y)$ is chosen at random (with uniform distribution) in the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Find the probability that, in the triangle with vertices at $(0, 0)$, $(1, 0)$ and p, the angle at each vertex is at most $\frac{5\pi}{12}$.

Solution: Let $A = (0, 0)$, $B = (1, 0)$, $C = (0, 1)$ and $D = (1, 1)$. Let E and F be the points on segment CD such that $\angle EAB = \frac{5\pi}{12} = \angle FBA$. Let G be the point on segment AC such that $\angle AGB = \frac{5\pi}{12}$ and let H be the point on segment BD such that $\angle BHA = \frac{5\pi}{12}$. Let S be the circle which passes through A, B, G and H. Let K and L be the points (other than A and B) which lie on S with A on the segment AE and L on the segment BF. Let O be the center of the circle. (You should draw the picture).

Let P be a point in the square $ABCD$, To have $\angle PAB \leq \frac{5\pi}{12}$, the point P must lie on or to the right of the segment AE. To have $\angle PBA \leq \frac{5\pi}{12}$, the point P must lie on or to the left of the segment BF. Since AB is a chord of the circle S, when P lies on S we have $\angle APB = \angle AGB = \angle AHB = \frac{5\pi}{12}$, so to have $\angle APB \leq \frac{5\pi}{12}$, the point P must lie on or outside of the circle S. Thus the required probability is equal to the area R of the region which lies between the segments AE and BF but outside the circle S. Let T be the area of trapezoid $KLEF$, let U be the area of triangle KOL, and let V be the area of the wedge which lies inside circle S between segments OK and OL. Then the required probability is

$$R = T + U - V\,.$$

Since L lies on the circle S, we have $\angle ALB = \frac{5\pi}{12}$. We also know that $\angle ABL = \frac{5\pi}{12}$, and so triangle LAB is isosceles with $|AL| = |AB| = 1$ and $\angle LAB = \frac{\pi}{6}$. It follows that $L = \left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. By symmetry we have $K = \left(1 - \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Note that on line AKE, A is the origin and the x-coordinate of K is $\frac{1}{2}$ of the x-coordinate of E, so we have $E = 2K = \left(2 - \sqrt{3}, 1\right)$. By symmetry, $F = \left(\sqrt{3} - 1, 1\right)$. Thus the trapezoid $KLEF$ has side lengths $|KL| = \sqrt{3} - 1$ and $|EF| = 2\sqrt{3} - 3$ and height $\frac{1}{2}$, so its area is

$$T = \tfrac{1}{4}\left(|KL| + |EF|\right) = \tfrac{1}{4}\left(3\sqrt{3} - 4\right).$$

Since $\angle KAB = \frac{5\pi}{12}$ and $\angle LAB = \frac{\pi}{6}$, we have $\angle KAL = \frac{5\pi}{12} - \frac{\pi}{6} = \frac{\pi}{4}$. Since KL is a chord of the circle S and O is the center, we have $\angle KOL = 2\angle KAL = \frac{\pi}{2}$. Thus triangle KOL is an isosceles right triangle. In this triangle, $|KL| = \sqrt{3} - 1$ and $|OK| = |OL| = \frac{1}{\sqrt{2}}|KL| = \frac{\sqrt{3}-1}{\sqrt{2}}$. Thus the area of triangle KOL is

$$U = \tfrac{1}{4}|KL|^2 = \tfrac{1}{4}(4 - 2\sqrt{3})\,.$$

Finally note that since $\angle KOL = \frac{\pi}{2}$, the wedge is one quarter of the circle of radius $|OK|$ so

$$V = \tfrac{1}{4}\,\pi\,|OK|^2 = \tfrac{2-\sqrt{3}}{4}\,\pi\,.$$

Thus the required probability is

$$R = T + U - V = \tfrac{\sqrt{3}}{4} - \tfrac{2-\sqrt{3}}{4}\,\pi\,.$$

5: Let x be an irrational number, and let M be a positive integer. Show that there exist integers a and b with $b > 0$ such that

$$\left|x - \tfrac{a}{b}\right| < \tfrac{1}{Mb}\,.$$

Solution: For a real number u, let $\langle u\rangle$ denote the fractional part of u, that is $\langle u\rangle = u - \lfloor u\rfloor$. Note that $\langle x\rangle \in [0,1)$. We divide $[0,1)$ into M equal subintervals

$$\left[0, \tfrac{1}{M}\right)\ ,\ \left[\tfrac{1}{M}, \tfrac{2}{M}\right)\ ,\ \left[\tfrac{2}{M}, \tfrac{3}{M}\right)\ ,\ \cdots\ ,\ \left[\tfrac{M-1}{M}, 1\right).$$

Two of the numbers $\langle x\rangle, \langle 2x\rangle, \langle 3x\rangle \cdots$ must lie in the same subinterval, say $k < l$ and $\langle kx\rangle$ and $\langle lx\rangle$ both lie in the same subinterval. Let $a = \big\lfloor (k-l)x \big\rfloor$ and $b = (k-l) > 0$. Then

$$bx - a = (k-l)x - \big\lfloor (k-l)x\big\rfloor = \big\langle (k-l)x\big\rangle = \langle kx\rangle - \langle lx\rangle \in \left[0, \tfrac{1}{M}\right)$$

and so $0 \le bx - a < \frac{1}{M}$ and hence $0 \le x - \frac{a}{b} < \frac{1}{Mb}$.

6: Let f be continuous on $[0,1]$ and differentiable in $(0,1)$. Suppose there exists $M > 0$ such that for all $x \in (0,1)$ we have $\big|f(0) - f(x) + xf'(x)\big| < Mx^2$. Prove that f is differentiable (from the right) at 0.

Solution: Let $g(x) = \dfrac{f(x) - f(0)}{x - 0}$. To show that f is differentiable at 0, we must show that $\lim\limits_{x\to 0^+} g(x)$ exists. We shall prove this by showing that $\lim\limits_{n\to\infty} g(x_n)$ exists for every sequence $\{x_n\}$ in $(0,1)$ with $x_n \to 0$.

We have $g'(x) = \dfrac{xf'(x) - f(x) + f(0)}{x^2}$. Since $\big|xf'(x) - f(x) + f(0)\big| < Mx^2$ for all $x \in (0,1)$, we see that $\big|g'(x)\big| < M$ for all $x \in (0,1)$. Now let $\{x_n\}$ be any sequence in $(0,1)$ with $x_n \to 0$. Let $\epsilon > 0$. Since $\{x_n\}$ is Cauchy, we can choose N so that for all integers $n, m \ge N$ we have $\big|x_n - x_m\big| < \frac{\epsilon}{M}$. Let $n, m \ge N$. By the Mean Value Theorem we can choose t between x_n and x_m so that $g'(t)(x_n - x_m) = g(x_n) - g(x_m)$. Then we have

$$\big|g(x_n) - g(x_m)\big| = |g'(t)||x_n - x_m| < M \cdot \frac{\epsilon}{M} = \epsilon\,.$$

Thus $\big\{g(x_n)\big\}$ is Cauchy, so it converges. (We remark that we did not need to use the hypothesis that f is continuous at 0).

Solutions to the Big E Problems, 2010

1: Find the minimum possible discriminant $\Delta = b^2 - 4ac$ of a quadratic $f(x) = ax^2 + bx + c$ which satisfies the requirement that $f\big(f\big(f(0)\big)\big) = f(0)$.

Solution: Let $f(x) = ax^2 + bx + c$. Then $f(0) = c$ so we have

$$\begin{aligned} f\big(f\big(f(0)\big)\big) = f(0) &\iff f\big(f(c)\big) = c \\ &\iff f(ac^2 + bc + c) = c \\ &\iff a(ac^2 + bc + c)^2 + b(ac^2 + bc + c) + c = c \\ &\iff (ac^2 + bc + c)\big(a(ac^2 + bc + c) + b\big) = 0 \\ &\iff (ac^2 + bc + c)(a^2c^2 + abc + ac + b) = 0 \\ &\iff c(ac + b + 1)(ac + 1)(ac + b) = 0 \\ &\iff c = 0\ , \quad ac = -(b + 1)\ , \quad ac = -1\ , \quad \text{or } ac = -b\,. \end{aligned}$$

When $c = 0$ we have $\Delta = b^2 - 4ac = b^2 \geq 0$, when $ac = -(b+1)$ we have $\Delta = b^2 + 4(b+1) = (b+2)^2 \geq 0$, when $ac = -1$ we have $\Delta = b^2 + 4 \geq 4$, and when $ac = -b$ we have $\Delta = b^2 + 4b = (b+2)^2 - 4 \geq -4$. Thus the minimum possible value for Δ is $\Delta = -4$, and this minimum value is attained when $b = -2$ and $ac = 2$, for example when $f(x) = x^2 - 2x + 2$.

2: Show that for every integer a, there exist infinitely many perfect powers of the form

$$a + 2010\,t\ ,\ \ t \in \mathbf{Z}\,.$$

(A *perfect power* is an integer of the form n^k for some integers $n \geq 0$ and $k \geq 2$).

Solution: Note that $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, which is a product of distinct primes. We claim, more generally, that if $m = p_1 p_2 \cdots p_l$ where the p_i are distinct primes, then for every $a \in \mathbf{Z}$ there exist infinitely many perfect powers of the form $a + mt$, $t \in \mathbf{Z}$. Let $\psi \geq 1$ be any common multiple of the numbers $p_i - 1$ for which $p_i \big| a$ (for example we could take $\psi = \phi(m)$). For those values of i for which $p_i \big| a$ we have $a \equiv 0 \pmod{p_i}$ and so $a^{\psi+1} \equiv 0 \equiv a \pmod{p_i}$. For those values of i for which $p_i \not| a$, by Fermat's Little Theorem we have $a^\psi \equiv 1 \pmod{p_i}$ and so again we have $a^{\psi+1} \equiv a \pmod{p_i}$. Thus $a^{\psi+1} \equiv a \pmod{p_i}$ for all $i = 1, 2, \cdots, l$, and so by the Chinese Remainder Theorem $a^{\psi+1} \equiv a \pmod{m}$. Finally note that for any $b \geq 0$ with $b \equiv a \pmod{m}$ we have $b^{\psi+1} \equiv a^{\psi+1} \equiv a \pmod{m}$, so we have found infinitely many perfect powers $b^{\psi+1}$ of the form $a + mt$, $t \in \mathbf{Z}$.

We remark that the above argument does not work when m has a factor of the form p^2 with p prime, and indeed when $p^2 \big| m$ there are no perfect powers of the form $p + mt$, $t \in \mathbf{Z}$.

3: Evaluate $\displaystyle\sum_{n=0}^{\infty}\int_0^{\pi}(-1)^n\sin^{2n}x\;dx$.

Solution: Using integration by parts, then replacing $\cos^2 x$ by $1-\sin^2 x$, we have

$$\begin{aligned}\int \sin^n x\;dx &= \int \sin^{n-1}x\;\sin x\;dx\\ &= -\sin^{n-1}x\;\cos x + \int (n-1)\sin^{n-2}x\;\cos^2 x\;dx\\ &= -\sin^{n-1}x\;\cos x + (n-1)\int \sin^{n-2}x\;dx - (n-1)\int \sin^n x\;dx\,.\end{aligned}$$

Adding $(n-1)\displaystyle\int \sin^n x\;dx$ to both sides then dividing by n gives

$$\int \sin^n x\;dx = -\tfrac{1}{n}\sin^{n-1}x\;\cos x + \tfrac{n-1}{n}\int \sin^{n-2}x\;dx$$

and so we have

$$\int_0^{\pi}\sin^n x\;dx = \tfrac{n-1}{n}\int_0^{\pi}\sin^{n-2}x\;dx\,.$$

This recursion formula gives $\displaystyle\int_0^{\pi}\sin^0 x\;dx=\pi$, $\displaystyle\int_0^{\pi}\sin^2 x\;dx=\tfrac12\,\pi$, $\displaystyle\int_0^{\pi}\sin^4 x\;dx=\tfrac34\cdot\tfrac12\cdot\pi$, and so on, so

$$\int_0^{\pi}\sin^{2n}x\;dx = \left(\tfrac12\cdot\tfrac34\cdot\tfrac56\cdots\tfrac{2n-1}{2n}\right)\pi = (-1)^n\,\frac{\left(-\frac12\right)\left(-\frac32\right)\left(-\frac52\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!}\,\pi = (-1)^n\binom{-1/2}{n}\pi\,.$$

By the Binomial Theorem and Abel's Theorem we have

$$\sum_{n=0}^{\infty}\int_0^{\pi}(-1)^n\sin^{2n}x\;dx = \pi\sum_{n=0}^{\infty}\binom{-1/2}{n} = \pi(1+1)^{-1/2} = \tfrac{\pi}{\sqrt2}\,.$$

4: Two points p and q are chosen at random (with uniform distribution) in the unit ball $x^2+y^2+z^2\le 1$. Find the probability that the triangle with vertices at p, q and the origin is an acute-angled triangle.

Solution: Let P_0 be the probability that the angle at 0 is at least $\frac{\pi}{2}$, let P_p be the probability that the angle at p is at least $\frac{\pi}{2}$, and let P_q be the probability that the angle at q is at least $\frac{\pi}{2}$. Since at most one of the three angles of triangle $0pq$ can be obtuse, the required probability P is equal to

$$P = 1 - P_0 - P_p - P_q\,.$$

We note that $P_0=\frac12$ since for each choice of p (with $p\neq 0$ if we wish to avoid a degenerate triangle), the angle at the origin is at least $\frac{\pi}{2}$ if and only if q lies in the half-ball given by $q\bullet p\le 0$ (with $q\neq tp$ for any t if we wish to avoid a degenerate triangle). We also note that $P_p=P_q$ by symmetry. Thus

$$P = \tfrac12 - 2P_q\,.$$

For $r>0$, the volume of spherical shell of radius r and infinitesimal thickness dr is $4\pi r^2\,dr$, so the probability that p lies in this shell is $\dfrac{4\pi r^2\,dr}{\frac43\pi}=3r^2\,dr$. Given that p lies in this shell, the points q for which the angle at q in the triangle $0pq$ is at least $\frac12$ are the points q which lie in or on the ball with diameter $0p$ (with $q\neq tp$ for any t if we wish to avoid a degenerate triangle). The volume of this ball is $\frac43\pi\left(\frac r2\right)^3$, so the probability that q lies in or on the ball is $\dfrac{\frac43\pi\left(\frac r2\right)^3}{\frac43\pi}=\frac18\,r^3$. Thus

$$P_q = \int_{r=0}^{1}\tfrac18 r^3\cdot 3r^2\,dr = \tfrac38\int_0^1 r^5\,dr = \tfrac38\cdot\tfrac16 = \tfrac1{16}$$

and hence $P=\frac12-2\cdot\frac1{16}=\frac38$.

5: Let A be the $n \times n$ matrix whose $(i,j)^{\text{th}}$ entry is $A_{i,j} = \frac{1}{i+j}$. Show that A is invertible.

Solution: We show that $\text{Null}(A) = 0$. Suppose that $Au = 0$ where $u = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{pmatrix}$ and let $f(x) = \sum_{k=0}^{n-1} u_k x^k$.

Let P_{n-1} denote the vector space of polynomials of degree at most $n-1$ with the inner product given by

$$\langle f, g \rangle = \int_0^1 x\, f(x) g(x)\, dx \,.$$

Notice that

$$Au = \begin{pmatrix} \frac{1}{2}u_0 + \frac{1}{3}u_1 + \cdots + \frac{1}{n+1}u_{n-1} \\ \frac{1}{3}u_0 + \frac{1}{4}u_1 + \cdots + \frac{1}{n+2}u_{n-1} \\ \vdots \\ \frac{1}{n+1}u_0 + \frac{1}{n+2}u_1 + \cdots + \frac{1}{2n}u_{n-1} \end{pmatrix} = \begin{pmatrix} \int_0^1 x f(x)\, dx \\ \int_0^1 x^2 f(x)\, dx \\ \vdots \\ \int_0^1 x^n f(x)\, dx \end{pmatrix} = \begin{pmatrix} \langle f, 1 \rangle \\ \langle f, x \rangle \\ \vdots \\ \langle f, x^{n-1} \rangle \end{pmatrix}.$$

Since $Au = 0$ we have $\langle f, 1 \rangle = \langle f, x \rangle = \cdots = \langle f, x^{n-1} \rangle = 0$. Since $\{1, x, \cdots, x^{n-1}\}$ is a basis for P_{n-1} it follows that $f \in {P_{n-1}}^{\perp} = \{0\}$, so $f = 0$ and hence $u = 0$.

6: Let f be continuous on $[0,1]$ and differentiable in $(0,1)$. Suppose there exists $M > 0$ such that for all $x \in (0,1)$ we have $\left| f(0) - f(x) + x f'(x) \right| < M x^2$. Prove that f is differentiable (from the right) at 0.

Solution: Let $g(x) = \dfrac{f(x) - f(0)}{x - 0}$. To show that f is differentiable at 0, we must show that $\lim_{x \to 0^+} g(x)$ exists. We shall prove this by showing that $\lim_{n \to \infty} g(x_n)$ exists for every sequence $\{x_n\}$ in $(0,1)$ with $x_n \to 0$.

We have $g'(x) = \dfrac{x f'(x) - f(x) + f(0)}{x^2}$. Since $\left| x f'(x) - f(x) + f(0) \right| < M x^2$ for all $x \in (0,1)$, we see that $\left| g'(x) \right| < M$ for all $x \in (0,1)$. Now let $\{x_n\}$ be any sequence in $(0,1)$ with $x_n \to 0$. Let $\epsilon > 0$. Since $\{x_n\}$ is Cauchy, we can choose N so that for all integers $n, m \geq N$ we have $\left| x_n - x_m \right| < \frac{\epsilon}{M}$. Let $n, m \geq N$. By the Mean Value Theorem we can choose t between x_n and x_m so that $g'(t)(x_n - x_m) = g(x_n) - g(x_m)$. Then we have

$$\left| g(x_n) - g(x_m) \right| = |g'(t)||x_n - x_m| < M \cdot \frac{\epsilon}{M} = \epsilon \,.$$

Thus $\{g(x_n)\}$ is Cauchy, so it converges. (We remark that we did not need to use the hypothesis that f is continuous at 0).