Maximal linear spaces contained in the base loci of pencils of quadrics

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Abstract

The geometry of the Fano variety of maximal linear spaces contained in the base locus of a pencil of quadrics has been studied by algebraic geometers when the base field is algebraically closed. In this paper, we work over an arbitrary base field of characteristic not equal to 2 and show how these Fano varieties are related to the Jacobians of hyperelliptic curves. In particular, if B is the base locus of a generic pencil of quadrics in $\mathbb{P}^{2n+1}$ and F is the Fano variety of $(n-1)$-planes contained in B, then F is a component of a disconnected commutative algebraic group $G = \mathrm{Pic}^0(C) \dot{\cup} F \dot{\cup} \mathrm{Pic}^1(C) \dot{\cup} F'$, where C is the hyperelliptic curve defined by the discriminant form of the pencil. In the second half of this paper, we study regular pencils of quadrics, where the hyperelliptic curve defined by the discriminant may be singular.

1. Introduction

Let k be a field of characteristic not 2. Let $\mathcal{L} = \{xQ_1 - yQ_2 \mid [x,y] \in \mathbb{P}^1\}$ be a pencil of quadrics in $\mathbb{P}^{N-1}$ for $N \geqslant 3$, where Q_1 and Q_2 are two linearly independent quadrics defined over k. Let $B = Q_1 \cap Q_2$ denote the base locus. In this paper, we study the general geometry of the variety F of linear subspaces contained in B of maximal dimension. Let A_1 and A_2 denote two Gram matrices of Q_1 and Q_2 with respect to the same choice of basis. Let $f(x)$ be the polynomial of degree at most N defined by

$$f(x) = \mathrm{disc}(xA_1 - A_2) = (-1)^{N(N-1)/2} \det(xA_1 - A_2).$$

We shall assume that $f(x)$ splits completely over a separable closure k^s of k.

The geometry depends very much on the parity of N and how "singular" the pencil is. A pencil is *generic* if $\mathcal{L}$ has transverse intersection with the hypersurface of singular quadrics in $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{N-1}}(2)))$. Equivalently, $\mathcal{L}$ contains precisely N singular quadrics over k^s, all of which are simple cones. This is also equivalent to saying that $f(x)$ has degree at least $N-1$ with no repeated roots. Suppose that $\mathcal{L}$ is generic. Denote by C the hyperelliptic curve with affine equation $y^2 = f(x)$. The isomorphism type of C over k is independent of the choice of the basis used to obtain the Gram matrices A_1 and A_2.

When $N = 2n+1$ is odd, the dimension of maximal linear subspaces contained in B is $n-1$. We assume that k has at least $N+1$ elements, which implies that there is a rational non-degenerate

quadric over k in the pencil. After renaming, we may assume that Q_1 is non-degenerate and hence the genus of C is n. It is well known from intersection theory that geometrically over k^s, there are 2^{2n} such $(n-1)$-planes. The arithmetic theory over k was studied in [BG14], where it was shown that F is a torsor of $J[2]$, where J denotes the Jacobian of C. Note that the isomorphism type of C over k depends on the choice of the two quadrics Q_1 and Q_2 in $\mathcal{L}$, while the group scheme $J[2]$ is independent of this choice.

When $N = 2n + 2$ is even, the theory is richer. The dimension of maximal linear subspaces contained in B is still $n - 1$, and the genus of C is still n. The rational function x defines a degree 2 morphism $C \to \mathbb{P}^1$. Let D_0 denote the hyperelliptic class obtained from pulling back the hyperplane section on $\mathbb{P}^1$. It was proved by Desale and Ramanan [DR76], Reid [Rei72], and Donagi [Don80] that geometrically over k^s, the variety F is isomorphic to the Jacobian J of C. Gauthier had also studied this in [Gau54]. The arithmetic theory when C has genus 1 is known and is used heavily in studying the 4-descent of elliptic curves. The genus 2 case was known to Cassels [Cas93]. The main result of this paper is that for arbitrary $n \geqslant 1$, the variety F is a torsor of J over k and, moreover, that the following theorem holds.

THEOREM 1.1. *Let G be the disconnected variety*

$$G = \mathrm{Pic}^0(C) \,\dot\cup\, F \,\dot\cup\, \mathrm{Pic}^1(C) \,\dot\cup\, F',$$

where F' is a copy of F. Then there is a commutative algebraic group structure on G over k such that

1. *$G^0 = \mathrm{Pic}^0(C)$ with component group $G/G^0 \simeq \mathbb{Z}/4$;*
2. *F' is isomorphic to F as varieties via the inversion map -1_G;*
3. *the group law extends that on $H = \mathrm{Pic}(C)/\mathbb{Z}D_0 \simeq \mathrm{Pic}^0(C) \,\dot\cup\, \mathrm{Pic}^1(C)$.*

Moreover, we will show that this structure is unique once we impose one more condition. See Theorem 2.25 for the complete statement.

When k is algebraically closed, Donagi put a group law on F in [Don80]. In light of Theorem 1.1, to put a group law on F amounts to choosing a point X_0 on $F(k)$ as the identity. The identity that Donagi picked is an element of $F[4] = \{X \in F \mid X +_G X +_G X +_G X = 0\}$. By studying the variety G and putting a canonical group structure on it, we no longer need to make this choice.

For arithmetic applications, it is of significance that Theorem 1.1 implies that as classes of torsors of J, we have $2[F] = [\mathrm{Pic}^1(C)]$ in $H^1(k, J)$. In particular, if $\mathrm{Pic}^1(C)(k) \neq \emptyset$, then F is a torsor of J of order dividing 2 and for any $[D_1] \in \mathrm{Pic}^1(C)(k)$, one has a lift of $[F]$ to a torsor of $J[2]$ by taking

$$F[2]_{[D_1]} = \{X \in F \mid X +_G X = [D_1]\}\,.$$

When the class $[D_1]$ comes from a rational point P on C, the lift $F[2]_P$ can also be described geometrically; see Examples 2.27 and 2.30.

When the curve C has a rational Weierstrass point P, it turns out that all torsors of $J[2]$ arise from pencils of quadrics as some $F[2]_P$. This is the key geometric input in [BG13], where Bhargava and Gross obtained the average sizes of the 2-Selmer groups of Jacobians of hyperelliptic curve with a rational Weierstrass point. When the curve C only has a rational non-Weierstrass point P, not all torsors of $J[2]$ arise this way in general. However, when k is a global field, all of the locally soluble torsors of $J[2]$, namely ones that correspond to elements of the 2-Selmer group, arise from pencils of quadrics. This is used in [SW18], where Shankar and the author obtained the average

sizes of the 2-Selmer groups of Jacobians of hyperelliptic curves with rational non-Weierstrass points.

In general, when restricted to F, the multiplication-by-2 map on G gives an unramified map from F to $\mathrm{Pic}^1(C)$ of degree 2^{2n}. Such a map is a twist of the multiplication-by-2 map from $\mathrm{Pic}^0(C)$ to itself and is called a two-cover of $\mathrm{Pic}^1(C)$. When $k = \mathbb{Q}$ is a global field and the curve C has points locally everywhere, all locally soluble two-covers of $\mathrm{Pic}^1(C)$ arise this way. By counting these locally soluble two-covers of $\mathrm{Pic}^1(C)$ and comparing them to locally soluble two-covers of $\mathrm{Pic}^0(C)$, Bhargava, Gross, and the author [BGW17] showed that $\mathrm{Pic}^0(C)$ and $\mathrm{Pic}^1(C)$ are non-isomorphic a positive proportion of the time and so the curve C has no points over any odd-degree extension of $\mathbb{Q}$ a positive proportion of the time.

In the second half of the paper, we generalize these results to the case where the pencil $\mathcal{L}$ is "slightly singular," or *regular*. A pencil $\mathcal{L}$ is regular if it only has simple cones as singular members. In other words, $\mathcal{L}$ could intersect tangentially to the discriminant hypersurface in $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{N-1}}(2)))$ but does not contain any quadrics with higher degeneracy degree than simple cones. Let U denote the underlying k-vector space of dimension N, and view the two generating quadrics Q_1, Q_2 as linear operators $A_1, A_2 \colon U \to U^\vee$. When Q_1 is smooth, A_1 is an isomorphism and the composite

$$T \colon U \xrightarrow{A_2} U^\vee \xrightarrow{A_1^{-1}} U$$

is self-adjoint with respect to A_1. We call T the self-adjoint operator associated to the pencil. The pencil spanned by Q_1 and Q_2 is regular if and only if T is regular, which by definition means that all the eigenspaces of T are 1-dimensional. Note that the pencil is generic if and only if T is regular semi-simple.

Suppose that $\mathcal{L}$ is regular with Q_1 non-degenerate and that $N = 2n + 1$ is odd. Factor $f(x)$ as $f(x) = c \prod_{i=1}^{r+1}(x - \alpha_i)^{m_i}$ over k^s. Let $U_{i,T}$ denote the generalized eigenspace over k^s of T with eigenvalue α_i. Since T is self-adjoint with respect to A_1, its generalized eigenspaces are pairwise orthogonal. A projective $(n-1)$-plane contained in the base locus B can be viewed as a linear n-plane X such that $X \subset X^\perp$ and $TX \subset X^\perp$, where $\perp$ is taken with respect to A_1. For each $i = 1, \ldots, r+1$, we define $\dim_{i,T}(X)$ to be the dimension of the maximal T-stable subspace of $(X \otimes k^s) \cap U_{i,T}$. Since each $U_{i,T}$ is m_i-dimensional and A_1 restricts to a non-degenerate quadratic form on $U_{i,T}$, we have

$$\dim_{i,T}(X) \leqslant m_i/2\,.$$

For any sequence of integers $d_1, \ldots, d_{r+1}$ such that $0 \leqslant d_i \leqslant m_i/2$, we define

$$L^T_{\{d_1,\ldots,d_{r+1}\}}(k^s) = \left\{ X \simeq (k^s)^n \mid X \subset X^\perp,\ TX \subset X^\perp,\ \dim_{i,T}(X) = d_i \right\}.$$

Note that the singular locus of B consists of the projectivization of the eigenspaces of T whose associated eigenvalues have multiplicity at least 2. Hence $L^T_{\{0,\ldots,0\}}(k^s)$ is the set of k^s-points of the variety of projective $(n-1)$-planes contained in the smooth locus of B. We have the following counting result on the size of $L^T_{\{d_1,\ldots,d_{r+1}\}}(k^s)$.

THEOREM 1.2. *Let a denote the number of d_i equal to $m_i/2$. Then*

$$\left| L^T_{\{d_1,\ldots,d_{r+1}\}}(k^s) \right| = 2^r/2^a\,.$$

Now, for any field k' containing k, one defines $L^T_{\{d_1,\ldots,d_{r+1}\}}(k')$ similarly. Let $J = \mathrm{Pic}^0(C)$ denote the generalized Jacobian of the (possibly singular) complete curve C defined by the affine equation $y^2 = f(x)$.

THEOREM 1.3. *For any field k' containing k, the group $J[2](k')$ acts on $L^T_{\{d_1,\dots,d_{r+1}\}}(k')$ simply transitively if $a = 0$ and transitively if $a > 0$.*

As before, the case when $N = 2n + 2$ is even is more interesting. In what follows, we write X for a linear subspace of U and write $\mathbb{P}X$ for its projectivization. For any $v \in U \otimes k^s$, denote by $[v]$ the point of $\mathbb{P}(U)(k^s)$ corresponding to the line spanned by v. As before, let C denote the hyperelliptic curve defined by the affine equation $y^2 = f(x)$. Then C is smooth if and only if the pencil is generic. When all the roots of f have multiplicity at most 2, the curve C is nodal. Denote by p_g the geometric genus of C, defined as the genus of the normalization $\widetilde{C}$. As we will see in Proposition 3.12, when $p_g = -1$, the curve C is reducible and the base locus B contains a unique $\mathbb{P}^n$. When $p_g \geqslant 0$, the base locus B contains no $\mathbb{P}^n$ and we define

$$F_0 = \left\{ \mathbb{P}X \mid \dim \mathbb{P}X = n - 1,\ \mathbb{P}X \subset B \right\}.$$

As in the odd case, we need to impose certain open conditions to obtain interesting relations with the Jacobian of C. Consider

$$F = \left\{ \mathbb{P}X \in F_0 \mid [v] \notin \mathbb{P}X \text{ for all singular points } [v] \in B \right\}.$$

Bhosle [Bho90] proved that when C only has nodal singularities, F is isomorphic to the generalized Jacobian of C over k^s. Over the base field k, we have the following result.

THEOREM 1.4. *Suppose $p_g \geqslant 0$ and that C only has nodal singularities. Let G be the disconnected variety*

$$G = \mathrm{Pic}^0(C) \,\dot\cup\, F \,\dot\cup\, \mathrm{Pic}^1(C) \,\dot\cup\, F',$$

where F' is a copy of F. There is a commutative algebraic group structure on G over k such that

1. *$G^0 = \mathrm{Pic}^0(C)$ with component group $G/G^0 \simeq \mathbb{Z}/4$;*

2. *F' is isomorphic to F as varieties via the inversion map -1_G;*

3. *the group law extends that on $H = \mathrm{Pic}(C)/\mathbb{Z}D_0 \simeq \mathrm{Pic}^0(C) \,\dot\cup\, \mathrm{Pic}^1(C)$, where D_0 is the hyperelliptic class.*

COROLLARY 1.5. *Over k^s, one can take F_0 as a compactification of the generalized Jacobian $\mathrm{Pic}^0(C)$.*

We expect Theorem 1.4 to be true without the nodal condition on C. Let $\widetilde{F}$ denote the torsor of $J(\widetilde{C})$ obtained from certain reductions (see Section 3) on the pencil $\mathcal{L}$ to the generic case. Without the condition on C, we can prove the following result.

THEOREM 1.6. *Suppose $p_g \geqslant 0$. Then there is a surjection $F \to \widetilde{F}$. Over k^s, the pre-image of every point has a filtration with $\mathbb{G}_a$- and $\mathbb{G}_m$-factors. The kernel of the natural map $J(C) \to J(\widetilde{C})$ has a filtration with the same factors.*

The theory of regular pencils is used in [BGW17] for the study of two-covers of $\mathrm{Pic}^1(C)$. As noted by Poonen and Schaefer in [PS97, footnote 2], it is not always enough to study only unramified covers of the hyperelliptic curve C when studying 2-descent on its Jacobian when C has no rational Weierstrass points; one also needs covers of C unramified away from the two points above some fixed point on $\mathbb{P}^1$. Identifying these two points gives a nodal curve, and regular pencils help in understanding two-covers for the generalized Jacobian of this nodal curve.

The paper is organized as follows. In Section 2, we focus on the case of generic pencils. We first recall from [BG14] the results in the case when N is odd. We then study the case when N

is even and prove Theorem 1.1. We finish with examples of special pencils in relation to the applications in [BG13] and [SW18]. In Section 3, we perform the parallel analysis in the case of regular pencils and prove Theorems 1.2–1.6.

2. Generic pencils

Let k be a field of characteristic not 2. Let Q_1 and Q_2 be two linearly independent quadratic forms on a k-vector space U of dimension N. In this section, we study the general geometry of the maximal isotropic subspaces with respect to both quadrics.

There are three equivalent ways to formulate this problem. We call the above formulation the (Q_1, Q_2)-setup. Now, suppose that Q_1 is non-degenerate. Let b_1 and b_2 denote the bilinear forms corresponding to Q_1 and Q_2, respectively. More explicitly,

$$b_i(v, w) = \frac{1}{2}\big(Q_i(v + w) - Q_i(v) - Q_i(w)\big).$$

Let $T \colon U \to U$ be the unique operator such that for all $v, w \in U$,

$$b_2(v, w) = b_1(v, Tw). \tag{2.1}$$

Note that T is self-adjoint with respect to b_1 since b_1 and b_2 are symmetric.

To say that a linear subspace X is isotropic with respect to both Q_1 and Q_2 is the same as saying

$$X \subset X^{\perp_{Q_1}}, \quad TX \subset X^{\perp_{Q_1}}. \tag{2.2}$$

Therefore, instead of starting with a pair of quadratic forms, we could have started with a non-degenerate quadratic form along with a self-adjoint operator. We call this formulation the (Q_1, T)-setup.

Lastly, we could view Q_1 and Q_2 as quadrics in $\mathbb{P}(U)$ and take a pencil $\mathcal{L} = \{xQ_1 - yQ_2 \mid [x, y] \in \mathbb{P}^1\}$ of quadrics in $\mathbb{P}(U)$. Let $B = Q_1 \cap Q_2$ denote the base locus. The above problem regarding common isotropic subspaces translates into studying the Fano variety of maximal linear subspaces contained in the base locus. We call this formulation the $(\mathbb{P}(U), \mathcal{L})$-setup.

We define the notion of *generic* in each of the three formulations. With the (Q_1, Q_2)-setup, we require $f(x) = (-1)^{N(N-1)/2} \det(xA_1 - A_2)$ to have no repeated roots, where A_1 and A_2 are Gram matrices for Q_1 and Q_2 with respect to the same choice of basis. With the (Q, T)-setup, we require the characteristic polynomial $f_T(x) = \det(xI - T)$ of the self-adjoint operator T to have no repeated roots. We will also assume that k has at least $N + 1$ elements, for otherwise there might not exist a rational non-degenerate Q in the pencil. With the $(\mathbb{P}(U), \mathcal{L})$-setup, we require that the pencil $\mathcal{L}$ intersects the hypersurface of singular quadrics in $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{N-1}}(2)))$ transversely. Equivalently, $\mathcal{L}$ contains precisely N singular quadrics over k^s, all of which are simple cones.

2.1 Odd dimension

We first consider the case when U has dimension $N = 2n+1$. Suppose that Q_1 is non-degenerate. Let C be the hyperelliptic curve of genus n defined by the affine equation

$$y^2 = f(x) = (-1)^n \det(xA_1 - A_2).$$

The isomorphism type of C over k is independent of the choice of the basis used for the Gram matrices A_1 and A_2. Let J denote the Jacobian of C. Suppose that the pencil is generic and $f(x)$

splits completely over k^s. Let $Q = Q_1$, and let T denote the corresponding self-adjoint operator. Let W_T denote the k-scheme of (linear) n-dimensional planes $X \subset U$ such that $X \subset X^\perp$ and $TX \subset X^\perp$.

The geometry over k^s is well known using classical intersection theory [Don80, Section 1.2]. We thank Elkies for the following explicit description of the 2^{2n} elements of $W_T(k^s)$. After a change of basis over k^s, one can assume that the two quadrics are given by

$$Q_1(x) = x_1^2 + x_2^2 + \cdots + x_{2n+1}^2 \,,$$
$$Q_2(x) = c_1 x_1^2 + c_2 x_2^2 + \cdots + c_{2n+1} x_{2n+1}^2$$

for some pairwise distinct $c_1, \ldots, c_{2n+1} \in k^s$. The following system of linear equations in $D_1, \ldots, D_{2n+1}$ has a 1-dimensional kernel:

$$D_1 + D_2 + \cdots + D_{2n+1} = 0 \,,$$
$$c_1 D_1 + c_2 D_2 + \cdots + c_{2n+1} D_{2n+1} = 0 \,,$$
$$\vdots$$
$$c_1^{2n-1} D_1 + c_2^{2n-1} D_2 + \cdots + c_{2n+1}^{2n-1} D_{2n+1} = 0 \,.$$

Choose a basis $(D_1, \ldots, D_{2n+1})$ for the kernel and note that $D_i \neq 0$ for all i. For each $i = 1, \ldots, 2n+1$, let d_i be a square root of D_i. Any choice of a system of square roots gives an element of $W_T(k^s)$ by taking

$$X = \{(d_1 P(c_1), \ldots, d_{2n+1} P(c_{2n+1})) \mid P \text{ is any polynomial of degree at most } n-1\} \,.$$

Note that if $(d_1 P(c_1), \ldots, d_{2n+1} P(c_{2n+1})) = (d_1' P'(c_1), \ldots, d_{2n+1}' P'(c_{2n+1}))$, where $d_i = \pm d_i'$ for all $i = 1, \ldots, 2n+1$ and P and P' are polynomials of degree at most $n-1$, then either $P + P'$ or $P - P'$ vanishes on at least $n+1$ elements out of $c_1, \ldots, c_{2n+1}$, implying that either $P = P'$, in which case $d_i = d_i'$ for all i, or $P = -P'$, in which case $d_i = -d_i'$ for all i. This shows that negating all of the d_i is the only way to produce the same n-plane. Hence we obtain 2^{2n} such n-planes from the different choices of square roots.

The arithmetic aspect of the theory has been studied by Bhargava and Gross in [BG14]; we list the results without proof. Consider the following two schemes over k:

$$V_f = \{T \colon U \to U \mid T \text{ is self-adjoint with respect to } Q$$
$$\text{with characteristic polynomial } f\} \subset \mathbb{A}^{(2n+1)^2} \,,$$
$$W_f = \{(T, X) \in V_f \times \mathrm{Gr}(n, U) \mid X \subset X^\perp \text{ and } TX \subset X^\perp\} \,.$$

Note that W_T is the fiber of W_f above a fixed T. The group $\mathrm{PO}(U, Q) = O(U, Q)/(\pm 1)$ acts on V_f and W_f via

$$g.T = gTg^{-1}, \quad g.(T, X) = \left(gTg^{-1}, gX\right) \,.$$

PROPOSITION 2.1. *Suppose that $f(x)$ splits completely in k^s with no repeated roots. Then the action of $\mathrm{PO}(U, Q)$ on V_f has a unique geometric orbit. For any $T \in V_f(k')$ defined over some field k' over k, its stabilizer scheme $\mathrm{Stab}(T)$ is isomorphic to $\mathrm{Res}_{L'/k'} \mu_2 / \mu_2 \simeq J[2]$ as group schemes over k', where $L' = k'[x]/f(x)$.*

For general Q, there might not be a self-adjoint operator defined over k with the prescribed characteristic polynomial. For example, over $\mathbb{R}$, operators self-adjoint with respect to a positive-definite form have real eigenvalues. This is not a problem if Q is split, that is, if there exists a linear space of dimension n over k isotropic with respect to Q.

PROPOSITION 2.2. *Suppose that $f(x)$ splits completely in k^s with no repeated roots. If Q is split over k, then $V_f(k)$ and $W_f(k)$ are non-empty. Furthermore, there exists $(T_0, X_0) \in W_f(k)$ with trivial stabilizer in $\mathrm{PO}(U, Q)(k^a)$, where k^a denotes an algebraic closure of k.*

THEOREM 2.3. *Suppose that k is separably closed of characteristic not 2, and suppose that $f(x)$ splits completely in k^s with no repeated roots. Then $\mathrm{PO}(U, Q)(k)$ acts simply transitively on $W_f(k)$.*

COROLLARY 2.4. *Suppose that k is arbitrary of characteristic not 2, and suppose that $f(x)$ splits completely in k^s with no repeated roots. If $W_f(k)$ is non-empty, then $\mathrm{PO}(U, Q)(k')$ acts simply transitively on $W_f(k')$ for any field k' over k.*

COROLLARY 2.5. *Suppose that k is arbitrary of characteristic not 2, and suppose that $f(x)$ splits completely in k^s with no repeated roots. Let $T \in V_f(k)$, and let J denote the Jacobian of the hyperelliptic curve defined by $y^2 = f(x)$. Then W_T is a torsor for $J[2]$.*

Remark 2.6. We now write down an explicit formula for the identification

$$J[2] \simeq \mathrm{Stab}(T). \tag{2.3}$$

We work over k^s, and it will be clear that the map is Galois-equivariant. Denote the roots of $f(x)$ over k^s by $\alpha_1, \ldots, \alpha_{2n+1}$, and the Weierstrass point corresponding to the root α_i by P_i for $i = 1, \ldots, 2n + 1$. Recall that $J[2]$ is an elementary 2-group generated by the $(P_i) - (\infty)$ with the only relation being that their sum is trivial. For each generator $(P_i) - (\infty)$, one looks for a polynomial $g_i(x)$ such that $g_i(\alpha_i) = -1$ and $g_i(\alpha_j) = 1$ for all $j \neq i$. Then $g_i(T)$ is the image of $(P_i) - (\infty)$ in $\mathrm{Stab}(T)$. The image does not depend on the choice of the polynomial g_i because any two choices differ by some multiples of $f(x)$ and $f(T) = 0$. More explicitly, we define $h_i(x) = f(x)/(x - \alpha_i)$ and take

$$g_i(x) = 1 - 2\frac{h_i(x)}{h_i(\alpha_i)}.$$

In other words, on the level of k^s-points, (2.3) is given by

$$\sum ((\alpha_i) - (\infty)) \mapsto \prod \left(1 - 2\frac{h_i(T)}{h_i(\alpha_i)}\right) = 1 - 2\sum \frac{h_i(T)}{h_i(\alpha_i)}.$$

The summations and product are written without indices, meaning that the equality holds for any (finite) collection of matching indices.

See Remark 2.29 for a different viewpoint of (2.3).

2.2 Even dimension

Next, consider the case when U has dimension $2n + 2$. The projective formulation is easier to work with in this case.

Let $\mathcal{L} = \{xQ_1 - yQ_2 \mid [x, y] \in \mathbb{P}^1\}$ be a rational generic pencil of quadrics in $\mathbb{P}^{2n+1} = \mathbb{P}(U)$. Rationality means that it is generated by two quadrics Q_1 and Q_2 defined over k. The cone points of the $2n + 2$ singular quadrics are best understood in terms of the self-adjoint operator T defined in (2.1), assuming that Q_1 is non-degenerate. The quadric $\lambda Q_1 - Q_2$ is singular if and only if λ is an eigenvalue of T. If we denote a corresponding eigenvector by v_λ, then the cone point of $\lambda Q_1 - Q_2$ is $[v_\lambda] \in \mathbb{P}(U)$. In particular, the $2n + 2$ cone points span the entire $\mathbb{P}(U)$.

Since $\mathcal{L}$ is generic, the maximal (projective) dimension of any linear space contained in the

base locus B is $n - 1$. Consider the following variety defined over k:

$$F = \{\mathbb{P}X \mid \dim(\mathbb{P}X) = n - 1, \mathbb{P}X \subset B\}.$$

The hyperelliptic curve C. For any rational generic pencil $\mathcal{L}$, there is an *associated hyperelliptic curve* defined as follows.

For any quadric Q in $\mathbb{P}^{2n+1}$, one defines its Lagrangian variety by

$$L_Q = \{\mathbb{P}Y \mid \mathbb{P}Y \subset Q, \dim(\mathbb{P}Y) = n\} \subset \mathrm{Gr}(n, \mathbb{P}(U)).$$

We recall some facts about L_Q; see also [Har92, Lecture 22]. When Q is smooth, L_Q has two connected components, also called the *rulings* of n-planes in Q. Two n-planes in Q lie in the same ruling if and only if their intersection codimension is even. If Q is defined over some field k', its *discriminant* is defined by

$$\mathrm{disc}(Q) = (-1)^{n+1}\det(Q)k'^{\times 2} \in k'^{\times}/k'^{\times 2}.$$

The connected components of L_Q are defined over $k'(\sqrt{\mathrm{disc}(Q)})$. In other words, $L_Q(k'^s)$ hits both rulings, and the $\mathrm{Gal}(k'^s/k'(\sqrt{\mathrm{disc}(Q)}))$-action on $L_Q(k'^s)$ preserves the rulings.[1] When Q is singular, L_Q has only one connected component.

Consider the variety

$$\widetilde{F} = \left\{(xQ_1 - yQ_2, \mathbb{P}Y) \mid [x,y] \in \mathbb{P}^1, \mathbb{P}Y \in L_{xQ_1-yQ_2}\right\} \subset \mathcal{L} \times \mathrm{Gr}(n, \mathbb{P}(U)).$$

There is an obvious projection map $p_1\colon \widetilde{F} \to \mathbb{P}^1$. The fiber over $[x,y] \in \mathbb{P}^1$ is isomorphic to $L_{xQ_1-yQ_2}$. Let

$$\epsilon\colon \widetilde{F} \to C, \quad \pi\colon C \to \mathbb{P}^1$$

denote the Stein factorization of p_1. In other words, ϵ has connected fibers, while the fibers of π correspond bijectively to the connected components of the fibers of p_1. Hence C is a double cover of $\mathbb{P}^1$ branched over the $2n + 2$ points that correspond to the singular quadrics on the pencil. A homogeneity analysis as in [Don80, Lemma 1.6] shows that C is smooth at the ramification points. Hence C is a hyperelliptic curve of genus n parameterizing the rulings in the pencil. We call C the hyperelliptic curve associated to the pencil. The Weierstrass points of C correspond to the $2n + 2$ points on $\mathbb{P}^1$ cut out by the binary form $\det(xQ_1 - yQ_2)$ of degree $2n + 2$. The curve C is isomorphic over k to the hyperelliptic curve defined by the affine equation

$$y^2 = (-1)^{n+1}\det(xQ_1 - Q_2),$$

canonically up to the hyperelliptic involution.

It was known to algebraic geometers [Rei72, DR76, Don80] that when k is separably closed, F is isomorphic to J, the Jacobian of the curve C defined above. Therefore, it is natural to expect that over a general field, F is a torsor of J.

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[1] We give a proof of this fact. Let Q be a non-degenerate quadric in $\mathbb{P}^{2n+1}$ over k'. Then by diagonalization, we may fix a basis $\{e_1, \dots, e_{2n+2}\}$ with respect to which Q is given by the quadratic form $a_1 x_1^2 + \cdots + a_{2n+2} x_{2n+2}^2$. The Lagrangian variety L_Q of Q is the disjoint union of two irreducible components. An element in $\mathrm{Gal}(k'^s/k')$ either fixes the components, or interchanges them. One may then take the isotropic (linear) $(n+1)$-plane $Y = \mathrm{Span}\{(1/\sqrt{a_1})e_1 + (1/\sqrt{-a_2})e_2, \dots, (1/\sqrt{a_{2n+1}})e_{2n+1} + (1/\sqrt{-a_{2n+2}})e_{2n+2}\}$ and observe that an element of $\mathrm{Gal}(k'^s/k')$ sends Y to an element in the same ruling if and only if it fixes an even number of the square roots $\sqrt{a_1}, \sqrt{-a_2}, \dots, \sqrt{a_{2n+1}}, \sqrt{-a_{2n+2}}$. The proof is now complete on noting that the discriminant of Q is $(-1)^{n+1}a_1 \cdots a_{2n+2} = a_1(-a_2) \cdots a_{2n+1}(-a_{2n+2})$.

A morphism $\tau\colon C \times F \to F$ that will serve as subtraction. Given any pair $(c, \mathbb{P}X) \in C \times F$, there is a unique n-plane $\mathbb{P}Y$ containing $\mathbb{P}X$ in the ruling of the quadric defined by c. Let Q be another quadric in the pencil. Since the base locus contains no n-planes, $\mathbb{P}Y \cap B = \mathbb{P}Y \cap Q$ is a quadric in $\mathbb{P}Y$ containing $\mathbb{P}X$. Hence $\mathbb{P}Y \cap B = \mathbb{P}X \cup \mathbb{P}X'$ is the union of two possibly equal $(n-1)$-planes. We define $\tau(c, \mathbb{P}X)$ to be $\mathbb{P}X'$. For any $c \in C$, define the involution $\tau(c)\colon F \to F$ by $\tau(c)\mathbb{P}X = \tau(c, \mathbb{P}X)$.

We can write down a more explicit formula for τ as follows. Given any $(c, \mathbb{P}X) \in C \times F$, let $\mathbb{P}Y$ denote the unique n-plane containing $\mathbb{P}X$ in the ruling of the quadric specified by c. Since $\mathbb{P}Y \not\subseteq Q$, there exists $p \in Y \backslash X$ such that $b(p, p) \neq 0$, where b is the bilinear form associated to Q. There is a linear map on $U \otimes k^s$ given by reflection about $p^{\perp_Q}$, namely

$$\mathrm{refl}_p\colon v \mapsto v - 2\frac{b(v, p)}{b(p, p)}p\,.$$

Then $\mathrm{refl}_p(X) \subset Y$ is isotropic with respect to Q and we have

$$\tau(c, \mathbb{P}X) = \mathbb{P}(\mathrm{refl}_p(X))\,.$$

Our goal is to put a natural group structure on $G = \mathrm{Pic}^0(C) \,\dot\cup\, F \,\dot\cup\, \mathrm{Pic}^1(C) \,\dot\cup\, F'$, where F' is a copy of F as algebraic varieties. We do so by defining a simply transitive action of $H = \mathrm{Pic}(C)/\mathbb{Z}D_0$ on $F \,\dot\cup\, F'$, for then we can define $+_G$ as follows: for any $x, x' \in F \,\dot\cup\, F'$, $[D], [D'] \in H$,

1. $[D] +_G [D']$ is the usual addition in H;
2. $x +_G [D] = x + [D]$ is the image of x under the action of $[D]$;
3. $x +_G x'$ is the unique element in H that sends $-x'$ to x.

An action of $\mathrm{Div}(C)$ on $F \,\dot\cup\, F'$. For any $x \in F$, write suggestively $-x$ for the corresponding element of F'. We start from the following action of C on $F \,\dot\cup\, F'$:

$$\mathbb{P}X + (c) = -\tau(\overline{c})\mathbb{P}X\,, \quad -\mathbb{P}X + (c) = \tau(c)\mathbb{P}X\,, \tag{2.4}$$

where $c \mapsto \overline{c}$ denotes the hyperelliptic involution. The second equality follows the idea that $\tau\colon C \times F \to F$ serves as a subtraction, and the first equality was rigged so that divisors linearly equivalent to the hyperellipitic class D_0 act trivially. Lemma 2.7 allows one to extend this action to the semi-group of effective divisors on C. Negating (2.4) then gives the extension to the entire group of divisors.

LEMMA 2.7. *For any $x \in F \,\dot\cup\, F'$ and $c_1, c_2 \in C$,*

$$(x + (c_1)) + (c_2) = (x + (c_2)) + (c_1)\,.$$

Proof. Unwinding the above definition, we need to prove that for any $\mathbb{P}X \in F$ and any $c_1, c_2 \in C$,

$$\tau(c_2)\tau(\overline{c_1})\mathbb{P}X = \tau(c_1)\tau(\overline{c_2})\mathbb{P}X\,.$$

As both sides are defined by polynomial equations, it suffices to prove this equality for generic $\mathbb{P}X$, c_1, c_2 over the algebraic closure; in particular, we may assume that no tangency is involved. This is proved in [Don80, Proof of Decomposition Lemma 2.6, p. 232] by looking at the intersection

$$\mathrm{Span}\{\mathbb{P}X, \tau(\overline{c_1})\mathbb{P}X, \tau(\overline{c_2})\mathbb{P}X\} \cap B\,. \qquad \square$$

THEOREM 2.8. *The above action of $\mathrm{Div}(C)$ descends to a simply transitive action of H on $F \,\dot\cup\, F'$.*

Remark 2.9. First, note that $+_G$ is defined over k, because τ is defined over k. Moreover, from the definition of the action of $\mathrm{Div}(C)$ on $F \mathbin{\dot{\cup}} F'$, we see that $[D]$ sends $-x'$ to x if and only if it sends $-x$ to x'. Hence $+_G$ is commutative.

Before proving Theorem 2.8, we give some concrete examples of $+_G$ in certain simple cases.

EXAMPLE 2.10. Suppose $n = 1$. Then F is the variety of points in the intersection of two generic quadrics in $\mathbb{P}^3$ and C is a genus 1 curve. Given two points $\mathbb{P}X, \mathbb{P}X' \in F$, let $\mathbb{P}Y$ denote the line passing through them. Then $\mathbb{P}Y$ lies on a unique ruling on a unique quadric in the pencil. Such data determines a point on C. Identifying C with $\mathrm{Pic}^1(C)$ naturally gives the map $+_G \colon F \times F \to \mathrm{Pic}^1(C)$.

EXAMPLE 2.11. Now, suppose that n is general and $\mathbb{P}X, \mathbb{P}X' \in F$ intersect in codimension 1. Let $\mathbb{P}Y = \mathrm{Span}\{\mathbb{P}X, \mathbb{P}X'\}$ denote their linear span. Then $\mathbb{P}Y \simeq \mathbb{P}^n$. Let p be a point on $\mathbb{P}Y \setminus (\mathbb{P}X \cup \mathbb{P}X')$. There is a quadric Q in $\mathcal{L}$ containing p. Its intersection with $\mathbb{P}Y$ contains two $\mathbb{P}^{n-1}$ and a point not on them. Hence Q contains the entire $\mathbb{P}Y$. Furthermore, since the pencil is generic, the base locus contains no $\mathbb{P}^n$. Hence $\mathbb{P}Y$ is contained in the unique quadric Q in $\mathcal{L}$ and a unique ruling on Q. Once again, such data determines a point on $c \in C$, and our group law says

$$\mathbb{P}X +_G \mathbb{P}X' = (c) \in \mathrm{Pic}^1(C) \,.$$

EXAMPLE 2.12. For any $\mathbb{P}X \in F$, since B is a complete intersection,

$$T_{\mathbb{P}X}B = T_{\mathbb{P}X}Q_1 \cap T_{\mathbb{P}X}Q_2 = \mathbb{P}\big(X^{\perp_{Q_1}} \cap X^{\perp_{Q_2}}\big) \,.$$

As Lemma 2.13 shows, $T_{\mathbb{P}X}B$ has dimension at most n. If $\mathbb{P}X \in F$ is such that $T_{\mathbb{P}X}B \simeq \mathbb{P}^n$, then just as in the above example, $T_{\mathbb{P}X}B$ lies on a unique ruling on a unique quadric in the pencil. Such data determines a point $c \in C$, and our group law says

$$\mathbb{P}X +_G \mathbb{P}X = (c) \in \mathrm{Pic}^1(C) \,.$$

As we will see in Example 2.27, for each Weierstrass point, there are 2^{2n} such $\mathbb{P}X$ for which $T_{\mathbb{P}X}B$ is isomorphic to $\mathbb{P}^n$ and is contained in the corresponding singular quadric.

LEMMA 2.13. *For a generic pencil $\mathcal{L}$, we have $\dim(T_{\mathbb{P}X}B) \leqslant n$.*

Proof. Suppose, without loss of generality, that Q_1 and Q_2 are non-degenerate. As $\dim(X) = n$, it follows that $\dim(X^{\perp_{Q_i}}) = n+2$ for $i = 1, 2$. Suppose, for a contradiction, $\dim(T_{\mathbb{P}X}B) \geqslant n+1$. Then $X^{\perp_{Q_1}} = X^{\perp_{Q_2}}$, and we denote this common linear $(n+2)$-plane by H. For any quadric Q in the pencil, we have $H \subset X^{\perp_Q}$. Since the cone points span the entire $\mathbb{P}(U)$, there exists a cone point $[v_\lambda]$ of a singular quadric $Q_\lambda \in \mathcal{L}$ such that $v_\lambda \notin H$. Since Q_λ descends to a quadratic form on the 2-dimensional vector space H/X, there exists a vector $v \in H \setminus X$ such that $Q_\lambda(v) = 0$. Now,

$$\mathrm{Span}\{X, v, v_\lambda\} \subset U$$

is an $(n+2)$-dimensional isotropic subspace with respect to Q_λ. However, since Q_λ is a simple quadric cone, its maximal isotropic subspace has dimension $n+1$, so we have a contradiction. $\square$

We will prove Theorem 2.8 by proving the following three propositions.

PROPOSITION 2.14. *The group $\mathrm{Div}(C)$ acts transitively on $F \mathbin{\dot{\cup}} F'$.*

PROPOSITION 2.15. *The principal divisors act trivially on $F \mathbin{\dot{\cup}} F'$. Since $[D_0]$ acts trivially, we now have a transitive action of H on $F \mathbin{\dot{\cup}} F'$.*

PROPOSITION 2.16. *If $[D] \in H$ acts trivially, then $[D] = 0$.*

Without loss of generality, we assume that k is algebraically closed. The following two lemmas proved in [Don80] are crucial in proving these propositions.

LEMMA 2.17 ([Don80, Lemma 2.6]). *Suppose that $\mathbb{P}X, \mathbb{P}X' \in F$ intersect at codimension $r \geqslant 1$. There exists a unique effective divisor D of degree r such that*

$$\mathbb{P}X + D = \mathbb{P}X' \text{ if } r \text{ is even}, \quad \mathbb{P}X + D = -\mathbb{P}X' \text{ if } r \text{ is odd}.$$

In particular, there exists $\mathbb{P}X'' \in F$ intersecting $\mathbb{P}X$ at codimension 1 while intersecting $\mathbb{P}X'$ at codimension $r - 1$.

LEMMA 2.18 ([Don80, Lemma 3.2]). *Suppose that D is an effective divisor with*

$$h^0(D) = \dim H^0(\mathcal{O}_C(D)) \geqslant 2,$$

where $H^0(\mathcal{O}_C(D)) = \{f \in k(C) \mid D + \operatorname{div}(f) \geqslant 0\}$. Then $[D] - [D_0]$ is also effective.

Proof of Proposition 2.14. It suffices to show that for any $\mathbb{P}X_1, \mathbb{P}X_2 \in F$, there exists an element $D \in \operatorname{Div}(C)$ sending $-\mathbb{P}X_1$ to $\mathbb{P}X_2$. Let $\mathbb{P}X_0$ be an element of F such that $T_{\mathbb{P}X_0}B$ is an n-plane. By Example 2.27, there are 2^{2n} such that $\mathbb{P}X_0$. Then by Example 2.12, there exists a point $e \in C$ such that $-\mathbb{P}X_0 + (e) = \mathbb{P}X_0$. We claim that for any $\mathbb{P}X \in F$, there exists a divisor $D \in \operatorname{Div}(C)$ sending $-\mathbb{P}X_0$ to $\mathbb{P}X$. Indeed, we induct on the intersection codimension r of $\mathbb{P}X_0$ and $\mathbb{P}X$. The base case $r = 0$ is when $\mathbb{P}X = \mathbb{P}X_0$, in which case $D = (e)$ does the job. Suppose that the claim is true for all $\mathbb{P}X'$ intersecting $\mathbb{P}X_0$ at codimension at most $r - 1$, and suppose that $\mathbb{P}X$ intersects $\mathbb{P}X_0$ at codimension r. By Lemma 2.17, there exists $\mathbb{P}X' \in F$ intersecting $\mathbb{P}X$ at codimension 1 while intersecting $\mathbb{P}X_0$ at codimension $r - 1$. By Example 2.11, there exists a point $e' \in C$ such that (e) sends $-\mathbb{P}X'$ to $\mathbb{P}X$. Let $D' \in \operatorname{Div}(C)$ be a divisor sending $-\mathbb{P}X_0$ to $\mathbb{P}X'$. It is then easy to see that the divisor $(e) - D' + (e')$ does the job.

Now, suppose that $\mathbb{P}X_1, \mathbb{P}X_2 \in F$ are arbitrary. Let D_1 and D_2 denote divisors sending $-\mathbb{P}X_0$ to $\mathbb{P}X_1$ and $\mathbb{P}X_2$, respectively. It is easy to see that the divisor $D_1 - (e) + D_2$ sends $-\mathbb{P}X_1$ to $\mathbb{P}X_2$. $\square$

LEMMA 2.19. *If $D \in \operatorname{Div}(C)$ fixes some $x_0 \in F \mathbin{\dot{\cup}} F'$, then D acts trivially.*

Proof. This follows immediately from the transitivity of the action and the commutativity of $\operatorname{Div}(C)$. $\square$

LEMMA 2.20. *If D and E are effective divisors of degree at most n and $D - E = \operatorname{div}(f)$ is a principal divisor, then $D - E$ acts trivially.*

Proof. Applying Lemma 2.18 repeatedly to D, one obtains a unique effective divisor D_1 with $h^0(D_1) = 1$ and such that D and E are in the linear system $D_1 + \frac{1}{2}(\deg(D) - \deg(D_1))D_0$. Since $\deg(D) \leqslant n$, the set $H^0\big(\mathcal{O}_C\big(\frac{1}{2}(\deg(D) - \deg(D_1))D_0\big)\big)$ consists entirely of functions pulled back from $\mathbb{P}^1$. Hence $D - E$ is a linear combination of divisors of the form $(P) + (\overline{P})$, which all act trivially on $F \mathbin{\dot{\cup}} F'$ by construction. $\square$

In what follows, we fix a Weierstrass point ∞ of C (defined over k^s).

LEMMA 2.21. *Suppose $D = (P_1) + \cdots + (P_r) - r(\infty) \in \operatorname{Div}(C)$ with $P_1, \ldots, P_r \neq \infty$ and $r \leqslant n$. If D is linearly equivalent to $E = (Q_1) + \cdots + (Q_{r'}) - r'(\infty)$ with $Q_1, \ldots, Q_{r'} \neq \infty$ and $r' \leqslant r$, then $x + D = x + E$ for all $x \in F \mathbin{\dot{\cup}} F'$.*

Proof. Apply Lemma 2.20 to $(P_1) + \cdots + (P_r)$ and $(Q_1) + \cdots + (Q_{r'}) + (r - r')(\infty)$. □

Every divisor class $[D] \in J = \mathrm{Pic}^0(C)$ can be represented by a divisor of the form $(P_1) + \cdots + (P_r) - r(\infty)$ with $P_1, \ldots, P_r \neq \infty$ and $r \leqslant n$. Lemma 2.21 says that two different representations of $[D]$ have the same action on $F \dot\cup F'$. Since $\deg(D)$ is even, it sends F to F. Hence we have a morphism of varieties

$$\alpha \colon J \to \mathrm{Aut}(F).$$

The image of α lies in a commutative subvariety of $\mathrm{Aut}(F)$. Since J is complete and $\alpha([0]) = \mathrm{id}$, rigidity [Mum70, II.4, pp. 40–41] implies that α is a group homomorphism.

Proof of Proposition 2.15. Let $\beta \colon \mathrm{Div}^0(C) \to \mathrm{Aut}(F)$ denote the action map. To show that the principal divisors act trivially, it suffices to show that β factors through $\alpha \colon J \to \mathrm{Aut}(F)$. Since both are group homomorphisms, it suffices to check

$$\beta((c) - (c')) = \alpha([(c) - (c')])$$

for any $c, c' \in C$. For any $\mathbb{P}X \in F$,

$$\begin{aligned}
\alpha([(c) - (c')])(\mathbb{P}X) &= \mathbb{P}X + (c) + (\overline{c'}) - 2(\infty) \\
&= \mathbb{P}X + (c) - (c') \\
&= \beta((c) - (c'))(\mathbb{P}X).
\end{aligned}$$
□

We now work toward the proof of Proposition 2.16. Given two elements $x = \pm\mathbb{P}X$ and $x' = \pm\mathbb{P}X'$ of $F \dot\cup F'$, we define their intersection codimension as the intersection codimension of $\mathbb{P}X$ and $\mathbb{P}X'$ and write

$$\mathrm{codim}(x, x') = \mathrm{codim}(\mathbb{P}X, \mathbb{P}X').$$

In this notation, Lemma 2.17 can be stated as follows.

LEMMA 2.22. *Suppose* $x, x' \in F$ *or* $x, x' \in F'$. *Then there exists a unique effective divisor D of degree $r = \mathrm{codim}(x, x')$ such that*

$$x + D = (-1)^r x'.$$

LEMMA 2.23. *Suppose that D is an effective divisor of degree r with $1 \leqslant r \leqslant n$. Then there exists $x \in F$ such that*

$$\mathrm{codim}(x, x + D) \equiv r \pmod 2.$$

There also exists $x \in F'$ satisfying the same condition.

Proof. The case when $r = 1$ follows immediately from Example 2.11. Now, suppose $r \geqslant 2$. Suppose, for a contradiction, that for all $x \in F$,

$$\mathrm{codim}(x, x + D) \equiv r - 1 \pmod 2. \tag{2.5}$$

Consider the closed variety

$$\Sigma = \{(x, c_1, \ldots, c_{r-1}) \mid x \in F,\ c_i \in C,\ x + D = -x + (c_1) + \cdots + (c_{r-1})\} \subset F \times \mathrm{Sym}^{r-1}(C).$$

Denote the two projections to F and $\mathrm{Sym}^{r-1}(C)$ by π_1 and π_2, respectively. We claim that π_1 is surjective. Indeed, let x be any element of F. Let r' denote the intersection codimension of x and $x + D$. Then $r' \leqslant r$. By Lemma 2.22, there exists an effective divisor D' of degree r' such that

$$x + D = (-1)^{r-r'}(x + D').$$

Assumption (2.5) says that $r - r'$ is odd for all x. Replacing D' by $D' + (r - 1 - r')(\infty)$ shows that π_1 is surjective. Since $\dim(F) = n$ and $\dim(\operatorname{Sym}^{r-1}(C)) = r - 1 < n$, there exists a fiber of π_2 of positive dimension. In other words, there exists a divisor $\widetilde{D}$ of odd degree such that for infinitely many $x \in F$,

$$x + \widetilde{D} = -x \,. \tag{2.6}$$

Let D_1 be a divisor such that $2D_1 - (\infty)$ is linearly equivalent to $\widetilde{D}$. Since we have shown that the principal divisors act trivially, (2.6) implies that for infinitely many $x \in F$,

$$(x + D_1) = -(x + D_1) + (\infty) \,.$$

Hence for infinitely many $\mathbb{P}X \in F$,

$$\mathbb{P}X = \tau(\infty)\mathbb{P}X \,.$$

However, as we will see in Example 2.27, there are only 2^{2n} such $\mathbb{P}X$, so we have a contradiction.

The statement for F' follows from the same argument, which is the main reason why we have used x to denote an element of F instead of the usual $\mathbb{P}X$. $\qquad\square$

Proof of Proposition 2.16. Suppose that $D = (P_1) + \cdots + (P_r) - r(\infty)$ acts trivially on F with $P_1, \ldots, P_r \neq \infty$ and $r \leqslant n$ minimal.

First, suppose that $r = 2r'$ is even. Then for all $\mathbb{P}X \in F$,

$$\mathbb{P}X + (P_1) + \cdots + (P_{r'}) = \mathbb{P}X + (\overline{P}_{r'+1}) + \cdots + (\overline{P}_r) \,. \tag{2.7}$$

By Lemma 2.23, there exists $\mathbb{P}X_0 \in F$ such that

$$\operatorname{codim}(\mathbb{P}X_0, \mathbb{P}X_0 + (P_1) + \cdots + (P_{r'})) = r'' \equiv r' \pmod 2 \,.$$

By the existence part of Lemma 2.17, there exist points $Q_1, \ldots, Q_{r''} \in C$ such that

$$\mathbb{P}X_0 + (P_1) + \cdots + (P_{r'}) = \mathbb{P}X_0 + (Q_1) + \cdots + (Q_{r''}) \,.$$

The divisor $(Q_1) + \cdots + (Q_{r''}) + (P_{r'+1}) + \cdots + (P_r) - (r'' + r')(\infty)$ then fixes $\mathbb{P}X_0$ and so acts trivially on F by Lemma 2.19. The minimality of r forces $r'' = r'$. That is,

$$\operatorname{codim}(\mathbb{P}X_0, \mathbb{P}X_0 + (P_1) + \cdots + (P_{r'})) = r' \,.$$

By the uniqueness part of Lemma 2.17, we have

$$(P_1) + \cdots + (P_{r'}) = (\overline{P}_{r'+1}) + \cdots + (\overline{P}_r)$$

as effective divisors of degree r'. Therefore, D is principal.

Now, suppose that $r = 2r' + 1$ is odd. Then for all $\mathbb{P}X \in F$,

$$\mathbb{P}X + (P_1) + \cdots + (P_{r'+1}) = \mathbb{P}X + (\overline{P}_{r'+2}) + \cdots + (\overline{P}_r) + (\infty) \,. \tag{2.8}$$

Arguing just like in the even case, we see that the minimality of r implies that for some $\mathbb{P}X_0 \in F$,

$$\operatorname{codim}(\mathbb{P}X_0, \mathbb{P}X_0 + (P_1) + \cdots + (P_{r'+1})) = r' + 1 \,.$$

Then Lemma 2.17 implies

$$(P_1) + \cdots + (P_{r'+1}) = (\overline{P}_{r'+2}) + \cdots + (\overline{P}_r) + (\infty)$$

as effective divisors of degree $r' + 1$. Therefore, D is principal. $\qquad\square$

We have completed the proofs of Propositions 2.14, 2.15, and 2.16. Before moving on to state the main theorem, we describe a stronger form of Lemma 2.23 for completeness.

Lemma 2.18 implies that if $(P_1)+\cdots+(P_r)-r(\infty)$ and $(Q_1)+\cdots+(Q_r)-r(\infty)$, with $r \leqslant n$, are two distinct divisors representing the same divisor class $[D] \in J$, then $[D]$ can also be represented by a divisor of the form $(R_1)+\cdots+(R_{r-2})-(r-2)(\infty)$. Hence if r is minimal among all such representations of $[D]$, there is a unique effective divisor D' of degree r such that

$$[D' - r(\infty)] = [D].$$

We call D' the ∞-minimal form of $[D]$.

COROLLARY 2.24. *Let D' be the ∞-minimal form of a non-zero divisor class $[D]$. Then there exists $x \in F$ such that*

$$\mathrm{codim}(x, x + D') = \deg(D').$$

There also exists $x \in F'$ satisfying the same condition.

Proof. Let r denote the degree of D'. Lemma 2.23 allows us to pick an $x \in F$ such that

$$\mathrm{codim}(x, x + D') =: r' \equiv r \pmod 2.$$

By Lemma 2.22, there exists an effective divisor D'' of degree r' such that $x + D' = x + D''$. Hence $D' - D''$ fixes x and by Lemma 2.19, $D' - D''$ acts trivially on F. By Proposition 2.16, D' is linearly equivalent to D''. Since D' is the ∞-minimal form of $[D]$, we see that $r' = r$.

The statement for F' follows from the same argument. $\qquad\square$

We now state our theorem in its complete form.

THEOREM 2.25. *Let G be the disconnected variety*

$$G = \mathrm{Pic}^0(C) \,\dot\cup\, F \,\dot\cup\, \mathrm{Pic}^1(C) \,\dot\cup\, F',$$

where F' is a copy of F. There is a unique commutative algebraic group structure $+_G$ on G over k such that

1. *$G^0 = \mathrm{Pic}^0(C)$ with component group $G/G^0 \simeq \mathbb{Z}/4$;*

2. *F' is isomorphic to F as varieties via the inversion map -1_G;*

3. *the group law extends that on $H = \mathrm{Pic}(C)/D_0 \simeq \mathrm{Pic}^0(C) \,\dot\cup\, \mathrm{Pic}^1(C)$, where D_0 is the hyperelliptic class;*

4. *the group law defines a simply transitive action of H on $F \,\dot\cup\, F'$ extending the following action of C:*

$$\mathbb{P}X + (c) = -\tau(c)\mathbb{P}X, \quad -\mathbb{P}X + (c) = \tau(\bar c)\mathbb{P}X,$$

   with respect to which $x +_G x'$, for $x, x' \in F \,\dot\cup\, F'$, is the unique divisor class sending $-x$ to x'.

Proof. The only thing left to check is the associativity, which amounts to the four equalities

$$[D_1] +_G ([D_2] +_G [D_3]) = ([D_1] +_G [D_2]) +_G [D_3],$$
$$x +_G ([D_2] +_G [D_3]) = (x +_G [D_2]) +_G [D_3],$$
$$x +_G (x' +_G [D_3]) = (x +_G x') +_G [D_3],$$
$$x +_G (x' +_G x'') = (x +_G x') +_G x''$$

for any $[D_1], [D_2], [D_3] \in H$ and any $x, x', x'' \in F \,\dot\cup\, F'$.

The first one is the associativity of the group law on H. The second follows from the definition of the action of H. The third follows as both sides send $-x$ to $x' + [D_3]$. For the fourth one,

denote the two sides by x_L and x_R and add x' to both sides. The third associativity tells us $x' +_G x_L = (x' +_G x) +_G (x' +_G x'')$ and likewise, $x_R +_G x' = (x +_G x') +_G (x'' +_G x')$. The commutativity of $+_G$ implies that these two elements of $\operatorname{Pic}^0(C)$ are equal. Therefore, $x_L = x_R$ is the image of $-x'$ under the action of $x' +_G x_L = x' +_G x_R$. $\qquad\square$

COROLLARY 2.26. *The class* $[F] \in H^1(k, J)$ *is 4-torsion, twice of which is* $[\operatorname{Pic}^1(C)]$. *One can lift* $[F]$ *to a torsor of* $J[4]$ *by taking*

$$F[4] := \{\mathbb{P}X \in F \mid \mathbb{P}X +_G \mathbb{P}X +_G \mathbb{P}X +_G \mathbb{P}X = 0\}.$$

When $\operatorname{Pic}^1(C)(k) \neq \emptyset$, the class $[F]$ is 2-torsion. For each $[D_1] \in \operatorname{Pic}^1(C)(k)$, we obtain a lift of F to a torsor of $J[2]$ by taking

$$F[2]_{[D_1]} = \{\mathbb{P}X \in F \mid \mathbb{P}X +_G \mathbb{P}X = [D_1]\}.$$

For the remainder of this section, we consider special cases where $[D_1] = (c)$ comes from a point c on the curve C.

2.2.1 *Example: Rational Weierstrass point*

EXAMPLE 2.27. Suppose that C has a rational Weierstrass point or, equivalently, that $\mathcal{L}$ has a rational singular quadric. By moving this point to ∞, we assume that Q_1 is singular with cone point $[v_\infty]$. Let $H = v_\infty^{\perp_{Q_2}}$ be the hyperplane in U orthogonal to v_∞ with respect to Q_2. Then $\tau(\infty)$ is induced by the linear map on U that fixes H and sends v_∞ to $-v_\infty$. Hence

$$F[2]_\infty = \{\mathbb{P}X \in F \mid \mathbb{P}X \subset B \cap \mathbb{P}H\}. \tag{2.9}$$

Notice that when restricted to the $(2n+1)$-dimensional vector space H, the two quadrics Q_1 and Q_2 span a generic pencil $\mathcal{L}_H$. Moreover, $Q_{1|H}$ is non-degenerate. Let T be the self-adjoint operator on H associated to the pencil $\mathcal{L}_H$ as defined in (2.1). Then the right-hand side of (2.9) is precisely W_T as defined in the odd-dimension case. Now $J[2]$ acts on $F[2]_\infty$ via the action of J on F and on W_T via the identification $J[2] \simeq \operatorname{Stab}_{\operatorname{PO}(H, Q_{1|H})}(T)$. It is not a priori clear that these two actions are the same, for the action of J on F is not induced by an action on the underlying vector space. The following result says that these two actions in fact coincide.

PROPOSITION 2.28. *As* $J[2]$-*torsors, we have* $F[2]_\infty = W_T$.

Proof. It suffices to show that for any $(P) - (\infty) \in J[2](k^s)$ with P a Weierstrass point, the two actions are the same. Let α denote the root of $f(x)$ corresponding to P, and set $h(x) = f(x)/(x - \alpha)$. On $W_T(k^s)$, by Remark 2.6, the action of $(P) - (\infty)$ is induced by the following map on $H \otimes k^s$:

$$x \mapsto x - 2\frac{h(T)}{h(\alpha)}x.$$

We now compute the action of $(P) - (\infty)$ on $F[2]_\infty(k^s)$. The singular quadric corresponding to P is $\alpha Q_1 - Q_2$. Let $w_P \in H \otimes k^s$ be an eigenvector of T with eigenvalue α. The cone point of $\alpha Q_1 - Q_2$ is $[(w_P, 0)]$. Here, we have decomposed $U \otimes k^s$ as $H \oplus U_{\infty, T}$, where $U_{\infty, T}$ is the kernel of the degenerate quadric Q_1. Let b denote the bilinear form associated to Q_1, and let b_1 denote the bilinear form associated to the restriction of Q_1 to H. From the definition of τ earlier, we see that the action of $(P) - (\infty)$ is induced by the following map on $U \otimes k^s$:

$$x \mapsto x - 2\frac{b(x, (w_P, 0))}{b((w_P, 0), (w_P, 0))}(w_P, 0).$$

If we view each $\mathbb{P}X \in F[2]_\infty(k^s)$ as sitting inside $\mathbb{P}H$, then the action of $(P) - (\infty)$ is induced by the following map on $H \otimes k^s$:

$$x \mapsto x - 2\frac{b_1(x, w_P)}{b_1(w_P, w_P)} w_P.$$

To prove Proposition 2.28, it remains to show that for any $x \in H \otimes k^s$, we have

$$\frac{h(T)}{h(\alpha)} x = \frac{b_1(x, w_P)}{b_1(w_P, w_P)} w_P.$$

Since both sides are killed by $T - \alpha$, and since T has 1-dimensional eigenspaces, they are both scalar multiples of w_P. Now

$$b_1\left(\frac{h(T)}{h(\alpha)} x, w_P\right) = b_1\left(x, \frac{h(T)}{h(\alpha)} w_P\right) = b_1(x, w_P) = b_1\left(\frac{b_1(x, w_P)}{b_1(w_P, w_P)} w_P, w_P\right).$$

Therefore, they are the same scalar multiple of w_P. □

Remark 2.29. Equation (2.9) offers another viewpoint for the canonical identification of $J[2]$ with the stabilizer of a self-adjoint operator: they share a common principal homogeneous space. Fix any k-rational T, and form the corresponding pencil of quadrics. Then $J[2]$ acts on $F[2]_\infty$ simply transitively and $\mathrm{Stab}(T)$ acts on W_T simply transitively. Identifying $F[2]_\infty$ and W_T by Proposition 2.28, we see from the definitions that these two actions commute. Fix some $X_0 \in F[2]_\infty$; one can define the map

$$\iota \colon J[2] \to \mathrm{Stab}(T)$$

by taking $\iota([D])$, for any $[D] \in J[2]$, to be the unique element of $\mathrm{Stab}(T)$ sending X_0 to $X_0 + [D]$. The commutativity of the two actions and the commutativity of $J[2]$ show that this map is independent of the choice of X_0. Proposition 2.28 then implies that ι is given by the map we defined in Remark 2.6.

2.2.2 *Example: Rational non-Weierstrass point*

EXAMPLE 2.30. Suppose that C has a rational non-Weierstrass point or, equivalently, that $\mathcal{L}$ has a rational quadric with discriminant 1. By moving this point to infinity, we assume that Q_1 has discriminant 1. The two rulings of Q_1 are then defined over k. Let Y_0 denote one of the rulings, and let $\infty \in C(k)$ denote the point corresponding to the quadric Q_1 and the ruling Y_0. Denote by ∞' the conjugate of ∞ under the hyperelliptic involution. Let T denote the self-adjoint operator on U associated to the pencil $\mathcal{L}$ as defined in (2.1). The next result gives a geometric interpretation for the torsor $F[2]_\infty$ of $J[2]$.

PROPOSITION 2.31.

$$F[2]_\infty = \{\mathbb{P}X \in F \mid \mathbb{P}X = \tau(\infty)\mathbb{P}X\} = \{\mathbb{P}X \simeq \mathbb{P}^{n-1} \mid \mathrm{Span}\{\mathbb{P}X, \mathbb{P}(TX)\} \sim Y_0\}.$$

The latter condition means that $\mathrm{Span}\{\mathbb{P}X, \mathbb{P}(TX)\} \simeq \mathbb{P}^n$ *is contained in* Q_1 *in the ruling* Y_0.

Proof. First, suppose that $\mathbb{P}X$ is an $(n-1)$-plane with $\mathrm{Span}\{\mathbb{P}X, \mathbb{P}(TX)\} \sim Y_0$. We show $\mathbb{P}X \in F[2]_\infty$. Indeed, since $\mathrm{Span}\{\mathbb{P}X, \mathbb{P}(TX)\}$ is isotropic with respect to Q_1, we have $TX \subset X^{\perp_{Q_1}}$ and hence $\mathbb{P}X \in F$. Since $\mathrm{Span}\{\mathbb{P}X, \mathbb{P}(TX)\} \supset \mathbb{P}X$ is an n-plane contained in Q_1 in the ruling Y_0, we see that $\tau(\infty)\mathbb{P}X$ is the residual intersection of $\mathrm{Span}\{\mathbb{P}X, \mathbb{P}(TX)\}$ with Q_2. Finally, $\mathrm{Span}\{\mathbb{P}X, \mathbb{P}(TX)\}$ intersects Q_2 tangentially at $\mathbb{P}X$ because

$$TX \subset TX^{\perp_{Q_1}} \Rightarrow TX \subset X^{\perp_{Q_2}} \Rightarrow T_{\mathbb{P}X} Q_2 \supset \mathrm{Span}\{\mathbb{P}X, \mathbb{P}(TX)\}.$$

Therefore, $\mathbb{P}X \in F[2]_\infty$.

Conversely, suppose $\mathbb{P}X \in F[2]_\infty$. Let $\mathrm{Span}\{\mathbb{P}X, [p]\}$ denote the n-plane contained in Q_1 in the ruling Y_0, for some $p \in U \otimes k^a$. We show

$$\mathrm{Span}\{\mathbb{P}X, \mathbb{P}(TX)\} = \mathrm{Span}\{\mathbb{P}X, [p]\}. \tag{2.10}$$

Let b_1 denote the bilinear form associated to Q_1. Since $\tau(\infty)\mathbb{P}X = \mathbb{P}X$, we have the following orthogonality conditions:

$$b_1(x, p) = b_1(x, Tp) = b_1(p, p) = 0 \quad \forall \, x \in X.$$

Since $\mathrm{Span}\{\mathbb{P}X, [p]\}$ does not lie in the base locus, we have $b_1(p, Tp) = Q_2(p) \neq 0$. Hence Tp is not orthogonal to p with respect to Q_1, implying $Tp \notin \mathrm{Span}\{X, p\}$. However, as $Tp \in X^{\perp Q_1}$, we have

$$X^{\perp Q_1} = \mathrm{Span}\{X, p, Tp\}.$$

Since $TX \subset p^{\perp Q_1} \cap X^{\perp Q_1}$, we have $TX \subset \mathrm{Span}\{X, p\}$. Hence to obtain the desired equality (2.10), it remains to rule out the possibility $TX \subset X$. Suppose, for a contradiction, $TX \subset X$. Then X, being T-stable, contains an eigenvector of T. Since T is regular semi-simple, its eigenvectors form an orthogonal basis with respect to Q_1. However, since Q_1 is non-degenerate, we see that none of the eigenvectors of T can be isotropic. We therefore have a contradiction. $\qquad\square$

In parallel to the odd-dimension case, Proposition 2.31 suggests fixing the monic polynomial $f(x)$ of degree $2n+2$ having no repeated roots and splitting completely over k^s, the quadratic form Q_1 of discriminant 1, and considering the following k-schemes:

$$V_f = \{T \colon U \to U \mid T \text{ is self-adjoint with characteristic polynomial } f(x)\},$$
$$W_f = \{(T, X) \in V_f \times \mathrm{Gr}(n, U) \mid \mathrm{Span}\{X, TX\} \sim Y_0\}.$$

Here $\mathrm{Span}\{X, TX\} \sim Y_0$ means that $\mathrm{Span}\{X, TX\}$ is a linear $(n + 1)$-plane isotropic with respect to Q_1 lying in the ruling Y_0. Let W_T denote the fiber of W_f above T. Proposition 2.31 then says that $F[2]_\infty = W_T$ as $\mathrm{Gal}(k^s/k)$-modules. In what follows, we show that W_T is also naturally a $J[2]$-torsor, where J is the Jacobian of the hyperelliptic curve C defined by the affine equation $y^2 = f(x)$, and that $F[2]_\infty = W_T$ as $J[2]$-torsors.

The group $\mathrm{SO}(U, Q_1)$ preserves the rulings. Hence it acts on W_f via

$$g.(T, X) = \left(gTg^{-1}, gX\right).$$

Since the central (± 1) acts trivially, we consider instead the action of the quotient $\mathrm{PSO}(U, Q_1) = \mathrm{SO}(U, Q_1)/(\pm 1)$.

PROPOSITION 2.32. *The action of* $\mathrm{PSO}(U, Q_1)$ *on* V_f *has a unique geometric orbit. For any* $T \in V_f(k')$ *defined over some field* k' *over* k, *its stabilizer scheme* $\mathrm{Stab}(T)$ *is isomorphic to* $(\mathrm{Res}_{L'/k'} \mu_2)_{N=1}/\mu_2 \simeq J[2]$ *as group schemes over* k', *where* $L' = k'[x]/f(x)$.

Proof. We prove the second statement first. Fix any T in $V_f(k')$. Since T is regular semi-simple, its stabilizer scheme in $\mathrm{GL}(U_{k'})$ is a maximal torus. It contains, and hence is equal to, the maximal torus $\mathrm{Res}_{L'/k'} \mathbb{G}_m$. For any k'-algebra K,

$$\mathrm{Stab}_{O(U_{k'}, Q_1)}(T)(K) = \{g \in (K[T]/f(T))^\times \mid g^*g = 1\}$$
$$= \{g \in (K[T]/f(T))^\times \mid g^2 = 1\}.$$

Hence

$$\mathrm{Stab}_{O(U_{k'},Q_1)}(T) \simeq \mathrm{Res}_{L'/k'}\,\mu_2\,,$$
$$\mathrm{Stab}_{SO(U_{k'},Q_1)}(T) \simeq (\mathrm{Res}_{L'/k'}\,\mu_2)_{N=1}\,,$$
$$\mathrm{Stab}_{PSO(U_{k'},Q_1)}(T) \simeq (\mathrm{Res}_{L'/k'}\,\mu_2)_{N=1}/\mu_2\,.$$

Suppose $T_1, T \in V_f(k^s)$. There exists $g \in GL(U)(k^s)$ such that $T_1 = gTg^{-1}$. Since T_1 and T are both self-adjoint, g^*g centralizes T and hence lies in $(k^s[T]/f(T))^\times$, which is a product of $k^{s\times}$ since f splits. Since the characteristic of k is not 2, there exists $h \in (k^s[T]/f(T))^\times$ such that $g^*g = h^2$. Then gh^{-1} is an element of $O(U,Q_1)(k^s)$ conjugating T to T_1. Multiplying h by $(-1, 1, \ldots, 1) \in (k^s[T]/f(T))^\times$ if necessary, we may assume $gh^{-1} \in SO(U,Q_1)(k^s)$. Its image in $PSO(U,Q_1)(k^s)$ does the job. $\square$

For general Q_1, just like in the odd-dimension case, there might not be a self-adjoint operator defined over k with the prescribed characteristic polynomial. The next result states that there is no obstruction when Q_1 is split.

LEMMA 2.33. *If Q_1 is split over k, then both $V_f(k)$ and $W_f(k)$ are non-empty. Furthermore, there exists $(T_0, X_0) \in W_f(k)$ with trivial stabilizer in $PSO(U,Q_1)(k^a)$.*

Proof. The main idea is to find a model space for the quadratic space (U, Q_1). Consider the $(2n+2)$-dimensional étale k-algebra $L = k[x]/f(x)$. Let β denote the image of x in L. Then $\{1, \beta, \beta^2, \ldots, \beta^{2n+1}\}$ forms a basis for L. On L, there is the following bilinear form:

$$\langle \lambda, \mu \rangle = \mathrm{Tr}(\lambda\mu/f'(\beta)) = \text{coefficient of } \beta^{2n+1} \text{ in } \lambda\mu\,.$$

This form defines a split quadratic form Q_0 since $Y = \mathrm{Span}_k\{1, \beta, \ldots, \beta^n\}$ is a rational isotropic subspace of dimension $n+1$. Hence there exists an isometry from (L, Q_0) to (U, Q_1) defined over k. Under this isometry, the multiplication-by-β operator maps to a self-adjoint operator, denoted by T_0, on L, while the n-plane $X = \mathrm{Span}_k\{1, \beta, \ldots, \beta^{n-1}\}$ maps to an n-plane, denoted by X_0, in L. Then $(T_0, X_0) \in W_f(k)$. Since $(\cdot\beta, X)$ has trivial stabilizer in $PSO(L, Q_0)(k^a)$, its image (T_0, X_0) has trivial stabilizer in $PSO(U, Q_1)(k^a)$. $\square$

THEOREM 2.34. *Suppose that k is separably closed of characteristic not 2. Then $PSO(U, Q_1)(k)$ acts simply transitively on $W_f(k)$.*

Proof. Proposition 2.32 shows that it suffices to prove that for the $T_0 \in V_f(k)$ obtained in Lemma 2.33, the group $\mathrm{Stab}(T_0)(k)$ acts simply transitively on $W_{T_0}(k)$. Since (T_0, X_0) has trivial stabilizer, it suffices to show that $W_{T_0}(k)$ and $\mathrm{Stab}(T_0)(k)$ have the same size. As a consequence of Proposition 2.31, for any k, the set $W_T(k^a) \simeq F[2]_\infty(k^a) \simeq J[2](k^a)$ has 2^{2n} elements for any T. Hence we are done, because

$$2^{2n} = |(\mathrm{Res}_{L/k}\,\mu_2/\mu_2)_{N=1}(k)| = |\mathrm{Stab}(T_0)(k)| \leqslant |W_{T_0}(k)| \leqslant |W_{T_0}(k^a)| = 2^{2n}\,. \qquad \square$$

COROLLARY 2.35. *Suppose that k is any field of characteristic not 2 and $W_f(k)$ is non-empty. Then $PSO(U, Q_1)(k')$ acts simply transitively on $W_f(k')$ for any field k' over k.*

Proof. It suffices to prove transitivity. Suppose that (T_1, X_1) and (T_2, X_2) are in $W_f(k')$. Let $g \in PSO(U, Q_1)(k'^s)$ be the unique element sending (T_1, X_1) to (T_2, X_2). Then for any $\sigma \in \mathrm{Gal}(k'^s/k')$, the element $^\sigma g$ also sends (T_1, X_1) to (T_2, X_2). Hence $g = {}^\sigma g$ and so $g \in PSO(U, Q_1)(k')$ by separable descent. $\square$

Remark 2.36. One can write down an explicit formula for the identification

$$\mathrm{Stab}(T) \simeq (\mathrm{Res}_{L/k}\,\mu_2)_{N=1}/\mu_2 \simeq J[2]\,. \tag{2.11}$$

The method is the same as the odd case in Remark 2.6. Denote the roots of $f(x)$ over k^s by $\alpha_1, \ldots, \alpha_{2n+2}$, and for each i, define $h_i(x) = f(x)/(x - \alpha_i)$. Then on the level of k^s-points, (2.11) is given by sending

$$\sum n_i(\alpha_i) - \frac{\sum n_i}{2}((\infty) + (\infty'))\,, \quad \sum n_i \text{ even}\,,$$

to the image in $\mathrm{PSO}_{2n+2}(k^s)$ of

$$\prod \left(1 - 2\frac{h_i(T)}{h_i(\alpha_i)}\right)^{n_i} = 1 - 2\sum n_i \frac{h_i(T)}{h_i(\alpha_i)}\,. \tag{2.12}$$

Just as in Proposition 2.28, the map $1 - 2h_i(T)/h_i(\alpha_i)$ is a reflection and so has determinant -1. The assumption that $\sum n_i$ is even ensures that the product in (2.12) lies in SO.

Just as in the odd case, $J[2]$ acts on $F[2]_\infty$ via the action of J and on W_T via the identification $J[2] \simeq \mathrm{Stab}_{\mathrm{PSO}(U,Q_1)}(T)$. The following result says that these two actions coincide.

PROPOSITION 2.37. *As $J[2]$-torsors, $F[2]_\infty = W_T$.*

Proof. It suffices to show that for any $(P_1) - (P_2) \in J[2]$, where P_1 and P_2 are any two Weierstrass points, the two actions are the same. Let α_i denote the root of $f(x)$ corresponding to P_i, and set $h_i(x) = f(x)/(x - \alpha_i)$ for $i = 1, 2$. On $W_T(k^a)$, by Remark 2.36, the action of $(P_1) - (P_2)$ is induced by the following map on $U \otimes k^s$:

$$x \mapsto x - 2\frac{h_1(T)}{h_1(\alpha_1)}x - 2\frac{h_2(T)}{h_2(\alpha_2)}x\,.$$

For $i = 1, 2$, let $w_i \in U \otimes k^s$ be an eigenvector of T with eigenvalue α_i. The cone point of the singular quadric corresponding to P_i is then $[w_i]$. Let b_1 denote the bilinear form associated to Q_1. Then on $F(k^s)$, the action of $\tau(P_i)$ is induced by the following map on $U \otimes k^a$:

$$\mathrm{refl}_{P_i} : x \mapsto x - 2\frac{b_1(x, w_i)}{b_1(w_i, w_i)}w_i\,.$$

Composing two such reflections, we see that the action of $\tau(P_1)\tau(P_2)$ is induced by the following map on $U \otimes k^a$:

$$x \mapsto x - 2\frac{b_1(x, w_1)}{b_1(w_1, w_1)}w_1 - 2\frac{b_1(x, w_2)}{b_1(w_2, w_2)}w_2 + \frac{4b_1(x, w_1)b_1(w_1, w_2)}{b_1(w_1, w_1)b_1(w_2, w_2)}w_2\,.$$

Since self-adjoint operators have pairwise orthogonal eigenspaces, the last term is 0. Also as in the proof of Proposition 2.28, for $i = 1, 2$,

$$\frac{h_i(T)}{h_i(\alpha_i)}x = \frac{b_1(x, w_i)}{b_1(w_i, w_i)}w_i\,.$$

Therefore, the two actions are equal. $\qquad\square$

Remark 2.38. In parallel to the odd case, the equality $F[2]_\infty = W_T$ as $\mathrm{Gal}(k^s/k)$-sets provides a different viewpoint on the identification of $J[2]$ with $\mathrm{Stab}(T)$, as they share a common principal homogeneous space. Proposition 2.37 implies that this new identification coincides with the formula given by Remark 2.36.

3. Regular pencils

For the rest of the paper, we focus on regular pencils. Let Ω_1 denote the discriminant hypersurface in $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{N-1}}(2)))$ parameterizing singular quadrics, and let Ω_2 denote the subvariety parameterizing quadrics with higher degeneracy degree. Recall that a pencil $\mathcal{L}$ is generic if and only if $\mathcal{L}$ is a generic line intersecting Ω_1 transversely at N points and missing Ω_2. A pencil is *regular* if it misses Ω_2 but is allowed to intersect Ω_1 tangentially.

In the (Q_1, T)-setup, where Q_1 is non-degenerate and T is self-adjoint with respect to Q_1, regularity of the pencil is equivalent to regularity of T. An operator T is regular if and only if its characteristic polynomial coincides with its minimal polynomial if and only if all of its eigenspaces are 1-dimensional. Let $f(x) = \det(xI - T)$ denote the minimal polynomial of T; as before, we assume that $f(x)$ splits completely over k^s.

The following reduction step is crucial to the study of the variety of maximal linear spaces contained in the base locus over k^s. Suppose, temporarily, that k is separably closed. Let U denote the underlying N-dimensional k-vector space. Let $v \in U$ denote an eigenvector of T whose eigenvalue α has multiplicity at least 2, and write $\langle v \rangle$ for the line spanned by v. Since v is an eigenvector and T is self-adjoint, T descends to a linear operator $\overline{T}$ on $\overline{U} = v^\perp/\langle v \rangle$, where $\perp$ is taken with respect to Q_1. The quadratic form Q_1 descends to a non-degenerate quadratic form $\overline{Q}_1$ on $\overline{U}$ with respect to which $\overline{T}$ is regular self-adjoint with minimal polynomial $f(x)/(x - \alpha)^2$. Suppose $N = 2n+1$ or $2n+2$. Let X be an n-plane in U such that $X \subset X^\perp$ and $TX \subset X^\perp$. Define $\overline{X}$ to be the image of $X \cap v^\perp$ in $v^\perp/\langle v \rangle$. Then $\overline{X} \subset \overline{X}^\perp$ and $\overline{TX} \subset \overline{X}^\perp$. As we will see in what follows, either $v \in X$ or $X \not\subseteq v^\perp$, and so $\dim \overline{X} = n - 1$. The strategy will be to apply this reduction repeatedly until T becomes regular semi-simple, in which case one can use the result in the previous section on generic pencils.

Factor $f(x)$ as $f(x) = \prod_{i=1}^{r+1}(x - \alpha_i)^{m_i}$ over k^s. Let $U_{i,T}$ denote the generalized eigenspace over k^s of T with eigenvalue α_i, and let v_i be an eigenvector of T with eigenvalue α_i. Regularity is equivalent to saying that each v_i is unique up to scalars. The singular locus of B consists precisely of all the $[v_i]$ with $m_i \geqslant 2$. For any linear n-plane X such that $X \subset X^\perp$ and $TX \subset X^\perp$ and for each $i = 1, \ldots, r+1$, we define $\dim_{i,T}(X)$ to be the dimension of the maximal T-stable subspace of $(X \otimes k^s) \cap U_{i,T}$. The generalized eigenspaces of T are pairwise orthogonal and so Q_1 restricts to a non-degenerate quadratic form on $U_{i,T}$. Since each $U_{i,T}$ is m_i-dimensional, we have

$$\dim_{i,T}(X) \leqslant m_i/2 \,.$$

For any sequence of integers $d_1, \ldots, d_{r+1}$ such that $0 \leqslant d_i \leqslant m_i/2$, we define for any field k' containing k,

$$L^{f,T}_{\{d_1,\ldots,d_{r+1}\}}(k') = \left\{ X \simeq (k')^n \mid X \subset X^\perp, TX \subset X^\perp, \dim_{i,T}(X) = d_i \right\}.$$

The superscript f is unnecessary, but it serves in making the reduction step clearer. Observe that $L^{f,T}_{\{0,0,\ldots,0\}}(k')$ is the set of k'-points of the variety of projective $(n-1)$-planes contained in the smooth locus of B.

3.1 Odd dimension

Suppose that $N = 2n+1$ is odd. For ease of notation, we write Q for the non-degenerate quadratic form Q_1. By multiplying Q by a constant, we also assume that Q has discriminant 1. Fix the minimal polynomial $f(x)$ of degree $2n+1$. Let C be the hyperelliptic curve defined by the affine

equation $y^2 = f(x)$. We define the k-scheme

$$V_f = \{T\colon U \to U \mid T \text{ is self-adjoint and regular with characteristic polynomial } f(x)\}.$$

Note that regularity is equivalent to requiring that there are no linear relations among 1, T, ..., T^{2n}. The same argument as in the proof of Lemma 2.33 gives that $V_f(k')$ is non-empty for any field k' over which Q is split.

For every field k' containing k and every $T \in V_f(k')$, let $W_T(k')$ denote the set of (linear) n-dimensional k'-subspaces X of $U \otimes k'$ such that $X \subset X^\perp$ and $TX \subset X^\perp$. As before, we define

$$W_f(k') = \{(T, X) \mid T \in V_f(k'), X \in W_T(k')\}.$$

There is a Galois-equivariant action of $\mathrm{PO}(U, Q) = O(U, Q)/(\pm 1)$ on W_f:

$$g.(T, X) = \left(gTg^{-1}, gX\right).$$

For any sequence of integers $d_1, \ldots, d_{r+1}$ such that $0 \leqslant d_i \leqslant m_i/2$, we define

$$W^f_{\{d_1, \ldots, d_{r+1}\}}(k') = \left\{(T, X) \mid T \in V_f(k'), X \in L^{f, T}_{\{d_1, \ldots, d_{r+1}\}}(k')\right\}.$$

The rest of the section is dedicated to proving the follow two theorems.

THEOREM 3.1. *Let a denote the number of d_i equal to $m_i/2$. Then*

$$\left|L^{f, T}_{\{d_1, \ldots, d_{r+1}\}}(k^s)\right| = 2^r/2^a.$$

The action of $\mathrm{PO}(U, Q)$ preserves the decomposition of $U \otimes k^s$ into generalized eigenspaces, in the sense that

$$U_{i, gTg^{-1}} = gU_{i, T} \quad \forall T \in V_f(k^s), \ \forall g \in \mathrm{PO}(U, Q)(k^s), \ \forall i = 1, \ldots, r+1.$$

Hence one obtains a Galois-equivariant action of $\mathrm{PO}(U, Q)(k^s)$ on $W^f_{\{d_1, \ldots, d_{r+1}\}}(k^s)$.

THEOREM 3.2. *The group $\mathrm{PO}(U, Q)(k^s)$ acts on $W^f_{\{d_1, \ldots, d_{r+1}\}}(k^s)$ simply transitively if $a = 0$ and transitively if $a > 0$.*

COROLLARY 3.3. *For any field k' over k such that $W^f_{\{0, \ldots, 0\}}(k')$ is non-empty, $\mathrm{PO}(U, Q)(k')$ acts simply transitively on $W^f_{\{0, \ldots, 0\}}(k')$.*

Proof. Given Theorem 3.2, the corollary follows by the same descent argument as in the proof of Corollary 2.35. $\square$

We begin by studying the conjugation action of $\mathrm{PO}(U, Q)$ on V_f and computing the stabilizers. This allows us to reduce Theorem 3.2 to two counting results: Theorems 3.1 and 3.6.

PROPOSITION 3.4. *The action of $\mathrm{PO}(U, Q)$ on V_f has a unique geometric orbit. For any $T \in V_f(k')$ defined over some field k' over k, its stabilizer scheme $\mathrm{Stab}(T)$ is isomorphic to $\mathrm{Res}_{L'/k'} \mu_2/\mu_2 \simeq J[2]$ as group schemes over k', where $L' = k'[x]/f(x)$. In particular, $\mathrm{Stab}_{\mathrm{PO}(U,Q)}(T)(k^s)$ is an elementary abelian 2-group of order 2^r, where $r + 1$ is the number of distinct roots $f(x)$ over k^s.*

Proof. The first statement follows in the same way as in the proof of Proposition 2.32, except that now, $k^s[x]/f(x)$ is a product of algebras of the form $k^s[x]/(x^{m_i})$. Every unit in $k^s[x]/(x^{m_i})$ is a square as $\mathrm{char}(k) \neq 2$.

The second statement follows from the structure theory of finitely generated modules over principal ideal domains. One can view $U \otimes k'$ as a module over $k'[x]$, with x acting via the

operator T. The elements in $\mathrm{GL}(U)(k)$ commuting with T are precisely the automorphisms of U as $k'[x]$-modules. Since T is regular, the structure theory of finitely generated modules over principal ideal domains says that $U \otimes k'$ is isomorphic to $k'[x]/f(x)$ as a $k'[x]$-module. As a module of $k'[x]$ generated by the element 1, the automorphisms of U are precisely multiplication by elements in $(k'[x]/f(x))^{\times}$. Then as in the proof of Proposition 2.32, we write $*$ for the operation of taking adjoint with respect to Q, and we have

$$\mathrm{Stab}_{O(U,Q)}(T)(k') = \{g(T) \mid g \in k'[x],\, g(T)^* g(T) = 1\}$$
$$= \mu_2(k'[T]^{\times}),$$
$$\mathrm{Stab}_{\mathrm{PO}(U,Q)}(T)(k') = \mu_2(k'[T]^{\times})/(\pm 1).$$

For the last statement, from the factorization of $f(x)$, we know

$$k^s[x]/f(x) \simeq \prod_{i=1}^{r+1} k^s[x]/(x-\alpha_i)^{m_i}.$$

Hence $\mathrm{Stab}_{O(U,Q)}(T)(k^s) \simeq (\mathbb{Z}/2\mathbb{Z})^{r+1}$ is an elementary abelian 2-group of order 2^{r+1}. Modding out the diagonally embedded $\mathbb{Z}/2\mathbb{Z}$ gives $\mathrm{Stab}_{\mathrm{PO}(U,Q)}(T)(k^s)$. $\square$

Remark 3.5. Just as in Remark 2.6, we can give a more explicit description for the stabilizer in terms of polynomials in T. For each $i = 1, \ldots, r+1$, define $h_i^T(x) = f(x)/(x-\alpha_i)^{m_i}$. Then

$$\mu_2(k^s[T]^{\times}) = \left\{ \prod_{i \in I} \left(1 - 2 \frac{h_i^T(T)}{h_i^T(\alpha_i)} \right) \right\}_{I \subset \{1,\ldots,r+1\}}$$
$$= \left\{ 1 - 2 \sum_{i \in I} \frac{h_i^T(T)}{h_i^T(\alpha_i)} \right\}_{I \subset \{1,\ldots,r+1\}}.$$

Note that for any $I \subset \{1, \ldots, r+1\}$ and any $j \notin I$, since $(x-\alpha_j)^{m_j}$ divides $h_i^T(x)$ in $k^s[x]$ and $(T-\alpha_j)^{m_j}$ kills the generalized eigenspace $U_{j,T}$, we see that

$$1 - 2 \sum_{i \in I} \frac{h_i^T(T)}{h_i^T(\alpha_i)}$$

acts trivially on $U_{j,T}$.

For any $T \in V_f(k^s)$, its stabilizer J_T in $\mathrm{PO}(U, Q)(k^s)$ acts on $L_{\{d_1,\ldots,d_{r+1}\}}^{f,T}(k^s)$. We rephrase Theorem 3.2 as follows.

THEOREM 3.6. *Let a denote the number of d_i equal to $m_i/2$. Then for any $X \in L_{\{d_1,\ldots,d_{r+1}\}}^{f,T}(k^s)$,*

$$|\mathrm{Stab}_{J_T}(X)| = 2^a.$$

Theorem 3.2 follows from Theorems 3.1 and 3.6 because the size of each orbit is

$$|J_T|/|\mathrm{Stab}_{J_T}(X)| = 2^r/2^a = \left| L_{\{d_1,\ldots,d_{r+1}\}}^{f,T}(k^s) \right|.$$

We will prove Theorems 3.1 and 3.6 via a series of reductions to the case of generic pencils.

Reduction on $d_1, \ldots, d_{r+1}$. Let X be any element of $L_{\{d_1,\ldots,d_{r+1}\}}^{f,T}(k^s)$ with $d_i \geqslant 1$ for some i. Let v_i denote an eigenvector of T corresponding to α_i. Since T is regular, v_i is unique up to

scaling. The assumption $d_i \geqslant 1$ then implies $v_i \in X$. Let b denote the bilinear form associated to Q. Note that we have $v_i \in v_i^{\perp}$ since there exists some v_i' such that $(T - \alpha_i)v_i' = v_i$, and hence

$$b(v_i, v_i) = b(v_i, (T - \alpha_i)v_i') = b((T - \alpha_i)v_i, v_i') = 0\,.$$

For any $w \in v_i^{\perp}$,

$$b(v_i, Tw) = b(Tv_i, w) = b(\alpha_i v_i, w) = 0\,.$$

Hence T descends to a linear map

$$\overline{T}_i \colon v_i^{\perp}/\langle v_i \rangle \to v_i^{\perp}/\langle v_i \rangle =: \overline{U}_i\,.$$

The quadratic form Q descends to a non-degenerate quadratic form $\overline{Q}_i$ with respect to which $\overline{T}_i$ is regular self-adjoint with characteristic polynomial $f(x)/(x-\alpha_i)^2$. Note that this reduction can be described projectively as intersecting the quadric defined by Q with the tangent plane to v_i, then projecting away from v_i.

Since $v_i \in X$ and X is isotropic, we see $X \subset v_i^{\perp}$. Let $\overline{X}_i$ denote the image of X in $\overline{U}_i$. It is immediate from the definition that $\overline{X}_i$ is $(n-1)$-dimensional, satisfying

$$\overline{X}_i \subset \overline{X}_i^{\perp \overline{Q}_i}\,, \quad \overline{T}_i \overline{X}_i \subset \overline{X}_i^{\perp \overline{Q}_i}\,,$$

and the dimensions of the maximal $\overline{T}_i$-stable subspaces in its intersection with the generalized eigenspaces are $d_1, \ldots, d_i - 1, \ldots, d_{r+1}$. We denote this reduction step by

$$\delta_i \colon L_{\{d_1, \ldots, d_{r+1}\}}^{f, T}(k^s) \xrightarrow{\sim} L_{\{d_1, \ldots, d_i - 1, \ldots, d_{r+1}\}}^{f/(x-\alpha_i)^2, \overline{T}_i}(k^s)\,.$$

Observe that δ_i is bijective with its inverse given by taking the pre-image of the projection map $v_i^{\perp} \to v_i^{\perp}/\langle v_i \rangle$.

How are the stabilizers affected by this reduction? If $h(x)$ is any polynomial in $k^s[x]$, then $\delta(h(T)X) = h(\overline{T}_i)\overline{X}_i$. Since δ is bijective, we conclude that $h(T)$ stabilizes X if and only if $h(\overline{T}_i)$ stabilizes $\overline{X}_i$. Note that if $m_i \geqslant 3$, then

$$h_i^T(x) = \frac{f(x)}{(x-\alpha_i)^{m_i}} = \frac{f(x)/(x-\alpha_i)^2}{(x-\alpha_i)^{m_i-2}} = h_i^{\overline{T}_i}(x)\,.$$

Hence according to the explicit description given in Remark 3.5,

$$h(T) \in J_T \iff h(\overline{T}_i) \in J_{\overline{T}_i}\,; \quad \text{hence} \quad |\operatorname{Stab}_{J_T}(X)| = |\operatorname{Stab}_{J_{\overline{T}_i}}(\overline{X}_i)|\,.$$

When $m_i = 2$ and $d_i = 1$, we no longer have α_i as an eigenvalue for $\overline{T}_i$. In this case,

$$J_T = \langle h(T), 1 - 2h_i(T)/h_i(\alpha_i) \mid h(\overline{T}_i) \in J_{\overline{T}_i} \rangle\,.$$

Now $U_{i,T} = \operatorname{Span}\{v_i, v_i'\}$ with $b(v_i, v_i') \neq 0$. Since $v_i \in X$ and X is isotropic, we have

$$X = \operatorname{Span}\{v_i, X \cap \operatorname{Span}\{U_{j,T}\}_{j \neq i}\}\,.$$

Now $1 - 2h_i(T)/h_i(\alpha_i)$ sends v_i to $-v_i$ and fixes every element in $\operatorname{Span}\{U_{j,T}\}_{j \neq i}$. Hence it stabilizes X and so

$$|\operatorname{Stab}_{J_T}(X)| = 2|\operatorname{Stab}_{J_{\overline{T}_i}}(\overline{X}_i)|\,.$$

Note that this case is precisely when a decreases by 1 in this reduction step.

We summarize this reduction step in the following proposition.

PROPOSITION 3.7. *Suppose* $d_i \geqslant 1$; *then there is a bijection*

$$\delta_i \colon L_{\{d_1, \ldots, d_{r+1}\}}^{f, T}(k^s) \xrightarrow{\sim} L_{\{d_1, \ldots, d_i - 1, \ldots, d_{r+1}\}}^{f/(x-\alpha_i)^2, \overline{T}_i}(k^s)\,.$$

When $m_i \geqslant 3$, the map δ_i does not change the sizes of the stabilizers. When $m_i = 2$, the map δ_i lowers the sizes of the stabilizers by a factor of 2.

Reduction on f. By the above reduction step, it remains to study $L^{f,T}_{\{0,0,\dots,0\}}(k^s)$. We do so by applying similar reduction steps to remove the repeated factors from $f(x)$. Most of the proofs in this subsection are purely linear algebra. We recommend the interested readers to prove these linear algebra statements themselves.

Suppose that α is a root of f of multiplicity $m \geqslant 2$. Let $X \in L^{f,T}_{\{0,0,\dots,0\}}(k^s)$ be arbitrary. Let v denote an eigenvector of T with eigenvalue α. As before, we can consider the descent to $\overline{U} = v^\perp/\langle v \rangle$. As in the above reduction step, Q descends to a non-degenerate quadratic form $\overline{Q}$ on $\overline{U}$, and T descends to a regular self-adjoint operator $\overline{T}$ with characteristic polynomial $f(x)/(x-\alpha)^2$.

Observe that we have $v \notin X$ since X contains no T-stable subspace. Hence the map $U \to U/\langle v \rangle$ is bijective when restricted to X. Consequently, $X \not\subseteq v^\perp$, for otherwise the $(2n-1)$-dimensional vector space $v^\perp/\langle v \rangle$ would contain an n-dimensional isotropic subspace, which is impossible. Now, $X \cap v^\perp$ has dimension $n-1$, and we denote its bijective image in $v^\perp/\langle v \rangle$ by $\overline{X}$.

PROPOSITION 3.8. *The above map sending X to $\overline{X}$ defines a surjection*

$$L^{f,T}_{\{0,0,\dots,0\}}(k^s) \longrightarrow L^{f/(x-\alpha)^2,\overline{T}}_{\{0,0,\dots,0\}}(k^s).$$

This map is bijective if $m \geqslant 3$ and is two-to-one if $m = 2$. In both cases,

$$|\operatorname{Stab}_{J_T}(X)| = |\operatorname{Stab}_{J_{\overline{T}}}(\overline{X})| \quad \text{for any } X \in L^{f,T}_{\{0,\dots,0\}}(k^s).$$

Proof. It is clear that $\overline{X}$ satisfies $\overline{X} \subset \overline{X}^\perp$ and $\overline{T}\,\overline{X} \subset \overline{X}^\perp$. We claim that $\overline{X}$ contains no $\overline{T}$-stable subspace. Since X contains no T-stable subspace, the only $\overline{T}$-stable subspace $\overline{X}$ could possibly contain is the line spanned by $v' + \langle v \rangle$, where v' is an element of U with $(T-\alpha)v' = v$. Suppose, for a contradiction, that X contains $v' + cv$ for some $c \in k^s$. Then $v = (T-\alpha)(v' + cv) \in X^\perp$, contradicting $X \not\subseteq v^\perp$. Hence $\overline{X} \in L^{f/(x-\alpha)^2,\overline{T}}_{\{0,0,\dots,0\}}(k^s)$.

Next, we prove surjectivity, compute the sizes of the fibers, and compare the sizes of the stabilizers. Suppose $\overline{X} \in L^{f/(x-\alpha)^2,\overline{T}}_{\{0,0,\dots,0\}}(k^s)$. Let b_α denote the bilinear form on U defined by

$$b_\alpha(u, u') = b(u, (T-\alpha)u').$$

The kernel of b_α is $\langle v \rangle$. Hence b_α descends to a non-degenerate bilinear form on the $(2n)$-dimensional vector space $U/\langle v \rangle$. Denote by $\perp_\alpha$ the operation of taking the perpendicular space with respect to b_α. Since $\overline{X}$ is $(n-1)$-dimensional, b_α further descends to a non-degenerate bilinear form on the 2-dimensional vector space $\overline{X}^{\perp_\alpha}/\overline{X}$. There are then two 1-dimensional isotropic lines. Denote by $\overline{X}_1$ and $\overline{X}_2$ their pre-images in $\overline{X}^{\perp_\alpha}$. The following lemma tells us how to "lift" to $L^{f,T}_{\{0,\dots,0\}}(k^s)$.

LEMMA 3.9. *Let $\overline{X}$ be an element of $L^{f/(x-\alpha)^2,\overline{T}}_{\{0,0,\dots,0\}}(k^s)$. Let $w \in U$ be an element such that $\overline{X}_w := \operatorname{Span}\{w + \langle v \rangle, \overline{X}\}$ is isotropic with respect to b_α. Suppose, further, $b(w, v) \neq 0$. Then there exists a unique $X^w \in L^{f,T}_{\{0,0,\dots,0\}}(k^s)$ such that the image of X^w under the map $U \to U/\langle v \rangle$ is $\overline{X}_w$.*

Proof. After scaling w, we assume $b(w, v) = 1$. Then $w - \frac{1}{2}b(w,w)v$ is the unique element of the form $w + cv$ such that $b(w + cv, w + cv) = 0$. Hence by replacing w with $w - \frac{1}{2}b(w,w)v$, we may

assume $b(w, w) = 0$. As a consequence, any lift of $\overline{X}_w$ must contain w. The following n-plane is then the only lift of $\overline{X}_w$ that is isotropic with respect to b:

$$X^w = \mathrm{Span}\{w, u - b(w, u)v\}_{u + \langle v \rangle \in \overline{X}} \subset U \,.$$

It is easy to see that we have $(T - \alpha)X^w \subset X^{w\perp}$. Since $w \notin v^\perp$, the n-plane X^w does not contain v and so contains no non-zero T-stable subspace. Hence $X^w \in L^{f,T}_{\{0,0,\dots,0\}}(k^s)$. $\qquad \square$

We first consider the case when $m \geqslant 3$. Let v'' be an element of U such that $(T - \alpha)v'' = v'$. Then

$$b_\alpha(v', v') = b(v', v) = b((T - \alpha)v'', v) = b(v'', (T - \alpha)v) = 0 \,.$$

Hence we may assume, without loss of generality, $\overline{X}_1 = \mathrm{Span}\{v' + \langle v \rangle, \overline{X}\} \subset v^\perp/\langle v \rangle$. Let w be an element of U such that $\overline{X}_2 = \mathrm{Span}\{w + \langle v \rangle, \overline{X}\}$. Since $\mathrm{Span}\{\overline{X}_1, \overline{X}_2\}$ has dimension $n + 1$, it is not isotropic with respect to b_α. Hence $b_\alpha(w, v') = b(w, v) \neq 0$. After scaling w, we may assume $b(w, v) = 1$. Since $b(v', v) = 0$, we cannot hope to lift $\overline{X}_1$ to some $X \not\subseteq v^\perp$. Lemma 3.9 implies that $\overline{X}_2$ admits a unique lift. This proves the surjectivity.

Now, suppose that $X' \in L^{f,T}_{\{0,\dots,0\}}(k^s)$ maps to $\overline{X}$. Then the image of X' in $U/\langle v \rangle$, denoted by $X'/\langle v \rangle$, is an n-plane isotropic to b_α, contains $\overline{X}$, and is b_α-orthogonal to $\overline{X}$. Since it does not contain $v' + \langle v \rangle$, we conclude that $X'/\langle v \rangle = \overline{X}_2$. The uniqueness part of Lemma 3.9 then shows that the fibers have size 1.

Just as in the previous reduction step, when $m \geqslant 3$, the stabilizers J_T and $J_{\overline{T}}$ consist of elements that are represented by the same set of polynomials in T and $\overline{T}$. It is clear that if $g(T)$ stabilizes X, then $g(\overline{T})$ stabilizes $\overline{X}$. Conversely, if $g(\overline{T})$ stabilizes $\overline{X}$, then $g(T)$ sends X to another n-plane that also maps to $\overline{X}$. Since there is only one such n-plane, we conclude that $g(T)$ also stabilizes X. Hence $|\mathrm{Stab}_{J_T}(X)| = |\mathrm{Stab}_{J_{\overline{T}}}(\overline{X})|$.

We now deal with the case $m = 2$. Write $\overline{X}_1 = \mathrm{Span}\{w_1 + \langle v \rangle, \overline{X}\}$ and $\overline{X}_2 = \mathrm{Span}\{w_2 + \langle v \rangle, \overline{X}\}$ for some $w_1, w_2 \in U$. We claim $w_1 \notin v^\perp$ and likewise $w_2 \notin v^\perp$. Suppose, for a contradiction, $w_1 \in v^\perp$; then $\overline{X}_1 \subset v^\perp/\langle v \rangle$. When $m = 2$, the quotient $v^\perp/\langle v \rangle$ is the orthogonal (with respect to b) direct sum of all the generalized eigenspaces not containing v and v'. Since $T - \alpha$ acts invertibly on generalized eigenspaces not containing v and v', we see that b_α descends to a non-degenerate bilinear form on $v^\perp/\langle v \rangle$. However, $\overline{X}_1$ is isotropic of dimension n while $v^\perp/\langle v \rangle$ has dimension $2n - 1$, so we have a contradiction.

We may now apply Lemma 3.9 to $\overline{X}_{w_1} = \overline{X}_1$ and $\overline{X}_{w_2} = \overline{X}_2$ to obtain two unique lifts X^{w_1} and X^{w_2}. Hence we have proved the surjectivity and showed that the fibers have size 2.

Regarding stabilizers, we are in the situation where compared to $J_{\overline{T}}$, the group J_T has an extra generator $h_0(T) = 1 - 2h(T)/h(\alpha)$, where $h(x) = f(x)/(x - \alpha)^2$. This extra generator fixes all the generalized eigenspaces not containing v. Hence $h_0(\overline{T})\overline{X}$ equals $\overline{X}$, and a simple computation shows that $h_0(\overline{T})\overline{X}$ switches $\overline{X}_1$ and $\overline{X}_2$. If $g(\overline{T})$ stabilizes $\overline{X}$, then either $g(T)$ stabilizes X^{w_1}, or it sends X^{w_1} to X^{w_2}, in which case $g(T)h_0(T)$ stabilizes X^{w_1}. Therefore, the sizes of the stabilizers remain unchanged. $\qquad \square$

COROLLARY 3.10. *We have $|L^{f,T}_{\{0,0,\dots,0\}}(k^s)| = 2^r$, and every element has trivial stabilizer in J_T.*

Proof. This follows from induction on the degree of f and the classical result on generic intersection in odd dimension recalled in Section 2.1. We write out the proof slightly differently from an induction argument so we can point out the differences between the contributions coming from roots of f with odd multiplicity and the contributions from roots with even multiplicity. Rewrite

the factorization of $f(x)$ as

$$f(x) = \prod_{i=1}^{s_1+1} (x - \beta_i)^{2n_i+1} \prod_{j=1}^{s_2} (x - \beta_j')^{2n_j'} ,$$

where each β_i is a root of $f(x)$ of odd multiplicity and each β_j' is a root of even multiplicity. Since $f(x)$ has odd degree, we know $s_1 \geqslant 0$ and $s_1 + s_2 = r$. Applying Proposition 3.8 repeatedly, one obtains the sequence of maps

$$L^{f,T}_{\{0,0,\ldots,0\}}(k^s) \xrightarrow{1 \text{ to } 1} L^{\prod_i(x-\beta_i)\cdot\prod_j(x-\beta_j')^2,T'}_{\{0,0,\ldots,0\}}(k^s) \xrightarrow{2^{s_2} \text{ to } 1} L^{\prod_i(x-\beta_i),T''}_{\{0,0,\ldots,0\}}(k^s) .$$

The last set has 2^{s_1} elements with trivial stabilizers, by the theory of generic pencils. Applying Proposition 3.8 again, one concludes that every element in $L^{f,T}_{\{0,0,\ldots,0\}}(k^s)$ has trivial stabilizer as well. The above diagram shows $\left| L^{f,T}_{\{0,0,\ldots,0\}}(k^s) \right| = 2^{s_1+s_2} = 2^r$. $\qquad \square$

Proof of Theorems 3.1 *and* 3.6. Applying Proposition 3.7 repeatedly gives a bijection

$$L^{f,T}_{\{d_1,\ldots,d_{r+1}\}}(k^s) \xrightarrow[\delta]{\sim} L^{\prod_i(x-\alpha_i)^{m_i-2d_i},T'}_{\{0,0,\ldots,0\}}(k^s)$$

such that for any $X \in L^{f,T}_{\{d_1,\ldots,d_{r+1}\}}(k^s)$,

$$|\operatorname{Stab}_{J_T}(X)| = 2^a |\operatorname{Stab}_{J_{T'}}(\delta(X))| .$$

This proves Theorem 3.6. The polynomial $g(x) = \prod_i (x - \alpha_i)^{m_i-2d_i}$ has $r + 1 - a$ distinct roots; hence applying Corollary 3.10 to g completes the proof of Theorem 3.1. $\qquad \square$

3.2 Even dimension

Now, suppose that U has dimension $N = 2n + 2$ for $n \geqslant 1$. As above, suppose that $Q = Q_1$ is non-degenerate, and denote by T the associated self-adjoint operator on U. Let $\perp$ denote the operation of taking perpendicular spaces with respect to Q. As in Section 2.2, let C be the (possibly singular) hyperelliptic curve parameterizing the rulings in the pencil. Then C is isomorphic over k, not canonically, to the hyperelliptic curve defined by

$$y^2 = (-1)^{n+1} \det(Q) \det(xI - T) .$$

The geometric genus p_g of C is defined to be the genus of its normalization $\widetilde{C}$. Let C^{sm} denote the smooth locus of C.

LEMMA 3.11. *Suppose $n \geqslant 0$. If W is an $(n+1)$-dimensional subspace of $U \otimes k^s$ isotropic with respect to Q_1 and Q_2, then W is T-stable.*

Proof. Since Q is non-degenerate, we have $W = W^{\perp_Q}$. Then $TW \subset W^{\perp_Q} = W$. $\qquad \square$

PROPOSITION 3.12. *The base locus B contains no $\mathbb{P}^n$ if and only if $p_g \geqslant 0$. When C is reducible or, equivalently, $p_g = -1$, the base locus B contains a unique $\mathbb{P}^n$.*

Proof. Without loss of generality, assume that k is algebraically closed. Suppose that B contains $\mathbb{P}W$ for some $(n+1)$-dimensional subspace W of U. Lemma 3.11 implies that W contains an eigenvector v of T. Since W is isotropic, the eigenvalue of v has multiplicity at least 2. One can now reduce the problem to $\overline{U} = v^\perp/\langle v \rangle$ and the n-dimensional subspace $\overline{W} = W/\langle v \rangle$. Apply Lemma 3.11 and reduce repeatedly until $\dim \overline{U} = 2$ and $\dim \overline{W} = 1$. Applying Lemma 3.11 again, we see that $\overline{T}$ has a repeated eigenvalue and hence that all the generalized eigenspaces

of T have even dimension, which implies that C is reducible. When C is reducible, $\overline{W}$ is the unique 1-dimensional eigenspace of $\overline{T}$, thus proving uniqueness. Existence follows from running the argument backward. $\qquad\square$

Let F_0 denote the following variety over k:

$$F_0 = \{\mathbb{P}X \mid \dim \mathbb{P}X = n-1,\, \mathbb{P}X \subset B\}.$$

In view of the Section 3.1 and Example 2.30, we impose an open condition and look at the following variety:

$$F = \{\mathbb{P}X \in F_0 \mid \operatorname{Span}\{X, TX\} \text{ has no non-zero } T\text{-stable subspace}\}. \qquad (3.1)$$

LEMMA 3.13. *Suppose $p_g \geqslant 0$. Then*

$$\begin{aligned}
F &= \{\mathbb{P}X \in F_0 \mid X \nsubseteq v^\perp \text{ for all singular points } [v] \in B\} \\
&= \{\mathbb{P}X \in F_0 \mid [v] \notin \mathbb{P}X \text{ for all singular points } [v] \in B\}.
\end{aligned}$$

Proof. Suppose $\mathbb{P}X \in F$. Let $[v]$ be any singular point of B. Since v is an eigenvector, $v \notin X$. Suppose, for a contradiction, $X \subset v^\perp$. Then $\mathbb{P}(\operatorname{Span}\{X, v\})$ is a $\mathbb{P}^n$ contained in B, contradicting Proposition 3.12. Hence $X \nsubseteq v^\perp$.

Conversely, suppose $\mathbb{P}X \in F_0 \backslash F$. Then $v \in \operatorname{Span}\{X, TX\}$ for some eigenvector v of T. Since X is isotropic with respect to every quadric in the pencil, we have $v \in \operatorname{Span}\{X, TX\} \subset X^\perp$. Hence $X \subset v^\perp$.

For the second equality, first suppose $X \subset v^\perp$ for some singular $[v] \in B$. Suppose, for a contradiction, $v \notin X$. Then after reducing to $v^\perp / \langle v \rangle$, the quotient $(X \cap v^\perp)/\langle v \rangle$ has dimension n, which contradicts Proposition 3.12. Hence $v \in X$. Conversely, if $v \in X$, then $X \subset v^\perp$ since X is isotropic. $\qquad\square$

Remark 3.14. The main reason why F was defined as in (3.1) instead of the more conceptual definitions in Lemma 3.13 is that there is still some interesting geometry when $p_g = -1$, as we will see toward the end of the paper, in which case (3.1) is the more appropriate definition.

As in the generic case, we work toward putting a group law on the disconnected variety

$$G = \operatorname{Pic}^0(C) \,\dot\cup\, F \,\dot\cup\, \operatorname{Pic}^1(C) \,\dot\cup\, F',$$

where F' is a copy of F. In what follows, we assume $p_g \geqslant 0$. Since the base locus contains no $\mathbb{P}^n$, one can define $\tau \colon C \times F_0 \to F_0$ as in the generic case.

LEMMA 3.15. *The map τ restricts to a morphism $C^{\mathrm{sm}} \times F \to F$.*

Proof. Recall that given a pair $(c, \mathbb{P}X) \in C^{\mathrm{sm}} \times F$, there is a unique $\mathbb{P}Y \simeq \mathbb{P}^n$ in the quadric and the ruling defined by c such that $\mathbb{P}X \subset \mathbb{P}Y$; then $\tau(c, \mathbb{P}X)$ is the residual intersection of $\mathbb{P}Y$ with the base locus. The claim here is $\tau(c, \mathbb{P}X) \in F$. Suppose, for a contradiction, $\mathbb{P}X' := \tau(c, \mathbb{P}X) \in F_0 \backslash F$. Then by Lemma 3.13, there exists a singular point $[v] \in B$ such that $v \in X'$. Since X and X' intersect at codimension 1 and $v \notin X$, we see that

$$\mathbb{P}Y = \operatorname{Span}\{\mathbb{P}X, \tau(c, \mathbb{P}X)\} = \operatorname{Span}\{\mathbb{P}X, [v]\}.$$

Let α denote the eigenvalue of v. Then since $\mathbb{P}X$ is isotropic with respect to every quadric in the pencil and v is in the kernel of the quadric Q_α, we see that $\mathbb{P}Y$ lies in Q_α. Hence $c = (\alpha, 0) \notin C^{\mathrm{sm}}$. We therefore have a contradiction. $\qquad\square$

As in the generic case, one obtains an action of C^{sm} on $F \,\dot\cup\, F'$:

$$\mathbb{P}X + (c) = -\tau(\bar{c})\mathbb{P}X\,, \quad -\mathbb{P}X + (c) = \tau(c)\mathbb{P}X\,. \tag{3.2}$$

This action extends to an action of $\mathrm{Div}(C^{\mathrm{sm}})$ on $F \,\dot\cup\, F'$. To show that this descends to a simply transitive action of $\mathrm{Pic}^0(C)$, we assume $k = k^a$ and work over the algebraic closure. Let v be an eigenvector with eigenvalue α of multiplicity $m \geqslant 2$. As usual, let $(\overline{U}, \overline{Q})$ denote the $2n$-dimensional quadratic space $v^\perp/\langle v\rangle$, and let $\overline{T}$ denote the descent of T to $\overline{U}$. Let $\overline{C}$ denote the (possibly singular) hyperelliptic curve

$$y^2 = \mathrm{disc}(\overline{Q})\det(xI - \overline{T}) = \mathrm{disc}(Q)\det(xI - T)/(x - \alpha)^2\,.$$

Note that $\overline{C} \to C$ is a partial normalization of C. There is a natural inclusion $\iota\colon C^{\mathrm{sm}} \hookrightarrow \overline{C}^{\mathrm{sm}}$. Define $\overline{B}$, $\overline{F}$, and $\overline{F}_0$ analogously to B, F, and F_0. Let $\mathbb{P}X$ be any element of F. Write, as before, $\overline{X} = (X \cap v^\perp)/\langle v\rangle$. Lemma 3.13 implies that $\overline{X}$ has the correct dimension. It is clear then that $\overline{X} \in \overline{F}_0$.

LEMMA 3.16. *The subspace* $\mathrm{Span}\{\overline{X}, \overline{T}\overline{X}\}$ *has no non-zero* $\overline{T}$-*stable subspace.*

Proof. Note that this is immediate when C has only nodal singularities, for this reduction step kills the α-generalized eigenspace and leaves the rest unchanged. In general, by Lemma 3.13, it suffices to show that $\overline{X}$ does not contain any singular point of $\overline{B}$. Let $v' \in U$ be such that $(T - \alpha)v' = v$. Suppose, for a contradiction, that $\overline{X}$ contains $v' + \langle v\rangle$. Then $v' + cv \in X$ for some $c \in k^s$. (Note that this can only happen if $m \geqslant 4$.) Hence $v = (T - \alpha)(v' + cv) \in X^\perp$, contradicting $\mathbb{P}X \in F$ by Lemma 3.13. Since the reduction step does not touch all the other generalized eigenspaces, we see that $\overline{X}$ does not contain any singular point of $\overline{B}$. $\qquad\square$

Denote this reduction step by $\delta_v\colon F \to \overline{F}$. We now have the following commutative diagram:

$$
\begin{array}{ccc}
C^{\mathrm{sm}} \times F & \longrightarrow & F \\
\downarrow{\scriptstyle \iota \times \delta_v} & & \downarrow{\scriptstyle \delta_v} \\
\overline{C}^{\mathrm{sm}} \times \overline{F} & \longrightarrow & \overline{F}\,.
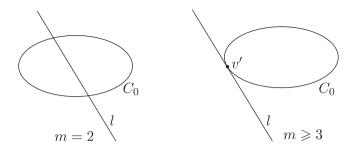\end{array}
$$

The natural map $\overline{C} \to C$ induces a map $J(C) \to J(\overline{C})$ on the Jacobians with kernel either $\mathbb{G}_m$ if $m = 2$, or $\mathbb{G}_a$ if $m \geqslant 3$. We now show that δ_v is surjective and that the pre-image of every point is isomorphic to $\ker(J(C) \to J(\overline{C}))$. Let b_α denote the bilinear form $b_\alpha(u, u') = b(u, (T - \alpha)u')$ on U, and let $\perp_\alpha$ denote the operation of taking the perpendicular space with respect to b_α. Fix any $\overline{X} \in \overline{F}$. The bilinear form b_α descends to a non-degenerate form on the $(2n+1)$-dimensional space $U/\langle v\rangle$. Inside this space, we have

$$\dim \overline{X}^{\perp_\alpha}/\overline{X} = 3\,,$$
$$\dim(\overline{X}^{\perp_\alpha} \cap v^\perp)/\overline{X} = 2\,.$$

In other words, b_α defines a smooth conic C_0 in $\mathbb{P}^2 = \mathbb{P}(\overline{X}^{\perp_\alpha}/\overline{X})$ and $l = \mathbb{P}((\overline{X}^{\perp_\alpha} \cap v^\perp)/\overline{X})$ is a line intersecting the conic at either one point or two points. The same argument as in Lemma 3.9 shows that every element in $C_0\backslash l$ admits a unique lift to an element of F. Our next result gives the characterization for the intersection behavior between C_0 and l.

LEMMA 3.17. *The line* l *intersects* C_0 *tangentially if and only if* $m \geqslant 3$, *in which case the point of intersection is* $[v' + \langle v\rangle + \overline{X}]$, *where* $v' \in U$ *is such that* $(T - \alpha)v' = v$.

$$m = 2 \qquad\qquad\qquad m \geqslant 3$$

Proof. Suppose that l intersects C_0 at a point $w + \langle v \rangle + \overline{X}$. To say that l intersects C_0 tangentially at $w + \langle v \rangle + \overline{X}$ is equivalent to saying

$$w + \langle v \rangle \in \overline{X}^{\perp_\alpha}, \quad b_\alpha(w, w) = 0, \quad w^{\perp_\alpha} \cap \overline{X}^{\perp_\alpha} = v^\perp \cap \overline{X}^{\perp_\alpha}. \tag{3.3}$$

Since $((v^\perp \cap \overline{X}^{\perp_\alpha})^{\perp_\alpha} \cap \overline{X}^{\perp_\alpha})/\overline{X}$ is the line spanned by $v' + \langle v \rangle + \overline{X}$, we see that the first and third conditions in (3.3) combined are equivalent to $w + \langle v \rangle + \overline{X} = v' + \langle v \rangle + \overline{X}$ up to scaling. Finally, $b_\alpha(v', v') = b(v', v) = 0$ if and only if $m \geqslant 3$. $\square$

Therefore, we have proved the following proposition.

PROPOSITION 3.18. *The map $\delta_v \colon F \to \overline{F}$ is surjective. The fibers are isomorphic to either (a conic minus a point $\simeq$) $\mathbb{G}_a$ when $m \geqslant 3$, or (a conic minus two points $\simeq$) $\mathbb{G}_m$ when $m = 2$. The kernel of the map $J(C) \to J(\overline{C})$ has the same property.*

One can now apply this reduction with any singular point of $\overline{B}$ and so on. For each i such that $m_i \geqslant 2$, let $v_{i,1}$ denote an eigenvector of T with eigenvalue α_i of multiplicity m_i, and let $v_{i,j}$ be such that $(T - \alpha_i)v_{i,j} = v_{i,j-1}$ for $j = 2, \ldots, \lfloor (m_i - 1)/2 \rfloor$. Let V denote the linear span of all such $v_{i,j}$. The above reduction will terminate at the $(2p_g + 2)$-dimensional vector space $\widetilde{U} = V^\perp/V$. The data (Q, T) descends to $(\widetilde{Q}, \widetilde{T})$ on the $(2p_g + 2)$-dimensional vector space $\widetilde{U} = V^\perp/V$, with $\widetilde{T}$ regular semi-simple. Let $\widetilde{F}$ denote the variety of (linear) p_g-dimensional common isotropic subspaces $\widetilde{X}$ contained in $\widetilde{U}$. Let $\delta \colon F \to \widetilde{F}$ denote the composite of all the reductions. The associated smooth hyperelliptic curve $\widetilde{C}$ is the normalization of C. Note that if k is arbitrary, then V is defined over k and so is the composite map δ. We summarize the above discussion into the following.

THEOREM 3.19. *Suppose $p_g \geqslant 0$ and that k is algebraically closed. Then:*

1. *The map $\delta \colon F \twoheadrightarrow \widetilde{F}$ is surjective. The pre-image of every point has a filtration with $\mathbb{G}_a$- and $\mathbb{G}_m$-factors. The kernel of the natural map $J(C) \to J(\widetilde{C})$ has a filtration with the same factors.*

2. *There is an action of $\mathrm{Div}^0(C^{\mathrm{sm}})$ on F that descends to the simply transitive action of $J(\widetilde{C})$ on $\widetilde{F}$.*

For the purpose of putting a natural group law on $G = \mathrm{Pic}^0(C) \,\dot\cup\, F \,\dot\cup\, \mathrm{Pic}^1(C) \,\dot\cup\, F'$, it remains to show that the action of $\mathrm{Div}(C^{\mathrm{sm}})$ on $F \,\dot\cup\, F'$ descends to a simply transitive action of $\mathrm{Pic}(C)/\mathbb{Z}D_0$ on $F \,\dot\cup\, F'$. Once again, we pass to the algebraic closure and use the same formal argument as in the generic case. We list the "non-formal" steps one needs to carry out in the regular case:

1. Prove Lemma 2.18, which allows one to define the ∞-minimal form of a divisor class $[D] \in J(C)$ and hence a morphism $\varphi \colon J \to \mathrm{Aut}(F)$. Here we need to assume that C has a smooth

Weierstrass point. This is equivalent to requiring that $f(x)$ has a simple root, or equivalently, $p_g \geqslant 0$.

2. Show that φ is a group homomorphism, to conclude that principal divisors supported on C^{sm} act trivially on $F \,\dot\cup\, F'$.

3. Prove the existence part of Lemma 2.17, to conclude that the action of J on F is transitive.

4. Prove the uniqueness part of Lemma 2.17, to conclude that the action is simply transitive.

Lemma 2.18 still holds in the singular case because the Riemann–Roch theorem holds in the singular case [Har77, IV.1, Exercise 1.9]. Suppose that C has a smooth Weierstrass point ∞. Note that if C only has nodal singularity and $p_g \geqslant 0$, then C has a smooth Weierstrass point. Every class $[D] \in J(C)$ has a ∞-minimal form $[D' - r(\infty)]$, where D' is effective of degree $r \leqslant n$, supported on C^{sm}, and $h^0(D') = 1$. This allows us to define a morphism of varieties $\varphi \colon J \to \mathrm{Aut}(F)$. The image of φ lies in a commutative subvariety of $\mathrm{Aut}(F)$.

We now specialize to the case when C only has nodal singularities, so J is an extension of an abelian variety $\widetilde{J}$ of dimension p_g by a torus S of dimension $n - p_g$.

LEMMA 3.20. *The map φ is a morphism of algebraic groups.*

Proof. The proof is very similar to the proof that a morphism between semi-abelian varieties mapping the identity to the identity is a group homomorphism. Without loss of generality, we may work over the algebraic closure. For any $s \in S$, its image in $\widetilde{J}$ is 0 and so by part 2 of Theorem 3.19, it acts on the fibers of the map $\delta \colon F \to \widetilde{F}$. For any $a \in J$, we define $\varphi_a \colon S \to \mathrm{Aut}(F)$ by

$$\varphi_a(s) = \varphi(a)\varphi(s)\varphi(as)^{-1} \,.$$

Fix any $x \in F$. Let S'_x denote the fiber of δ over $\delta(x)$. Then S'_x is a torus by setting x as the origin. For any $a \in J$ and any $s \in S$, we have $\delta(x) = \delta(\varphi_a(s)(x))$ by part 2 of Theorem 3.19. We have then defined a map $\varphi_{a,x} \colon S \to S'$ between tori sending s to $\varphi_a(s)(x)$. Since $\varphi_{a,x}$ sends the origin of S to the origin x of S'_x, we see that $\varphi_{a,x}$ is a group homomorphism. Letting a vary, one obtains a map $\Phi_x \colon J \to \mathrm{Hom}(S, S'_x)$ sending a to $\varphi_{a,x}$. Since J is connected and $\mathrm{Hom}(S, S'_x)$ is discrete, Φ_x is constant. Hence for any $a \in J$, the map $\Phi_x(a) = \Phi_x(0)$ is the trivial map $S \to S'_x$. Letting x vary, we have proved that

$$\varphi(a)\varphi(s) = \varphi(as) \quad \forall\, a \in J \,, \ s \in S \,. \tag{3.4}$$

Now, fix $a \in J$ and view φ_a as a morphism $J \to \mathrm{Aut}(F)$. Since φ_a vanishes on S by (3.4), we descend φ_a to a morphism $\widetilde{J} \to \mathrm{Aut}(F)$. Once again by part 2 of Theorem 3.19, fixing any $x \in F$, $\varphi_a(a')$ acts on the fiber over $\delta(x)$ for any $a' \in \widetilde{J}$. Hence we have a morphism $\varphi_{a,x} \colon \widetilde{J} \to S'_x$, sending a' to $\varphi_a(a')(x)$, which is trivial since $\widetilde{J}$ is an abelian variety and S'_x is a torus. Letting x vary, one sees that φ_a is trivial. Letting a vary gives the desired result. $\square$

As in the proof of Proposition 2.15, we have shown that principal divisors supported on C^{sm} act trivially on $F \,\dot\cup\, F'$. Next, we show the transitivity of this action. Since $\mathrm{Div}(C^{\mathrm{sm}})$ also acts on $F_0 \,\dot\cup\, F'_0$ and $F \,\dot\cup\, F'$ is open in $F_0 \,\dot\cup\, F'_0$, by taking the Zariski closure, one sees that principal divisors supported on C^{sm} act trivially on $F_0 \,\dot\cup\, F'_0$. Since being supported on C^{sm} is also an open condition, one also has that principal divisors on C act trivially on F_0. The existence part of Lemma 2.17 can be applied to F_0 since the defining map $C \to \mathbb{P}^1$ admits no section. As a result, given $x, x' \in F$, view them as in F_0, where there exists an effective divisor $D \in \mathrm{Div}(C)$ such that $x + D = \pm x'$. Let D' be a divisor supported on C^{sm} linearly equivalent to D. Since principal divisors on C act trivially, $x + D' = x + D = \pm x'$. Transitivity then follows from the formal

argument in the proof of Proposition 2.14. Note that here, the existence of a smooth Weierstrass point is needed because we need to know that there exists $\mathbb{P}X_0 \in F$ such that $T_{\mathbb{P}X_0}B \simeq \mathbb{P}^n$.

The uniqueness part of Lemma 2.17 also holds for F_0. The argument in [Don80] works since there is no injective map from $\mathbb{P}^1$ to C when the arithmetic genus n of C is at least 1. The same formal argument in the generic case then implies that only principal divisors act trivially. Once again, the existence of a smooth Weierstrass point ∞ is also needed, for we need the analog of Example 3.24 to know that there are finitely many elements of F fixed by $\tau(\infty)$. By combining this with all the formal arguments in Section 2.2, we have now finished putting a natural group law on $G = \operatorname{Pic}^0(C) \mathbin{\dot\cup} F \mathbin{\dot\cup} \operatorname{Pic}^1(C) \mathbin{\dot\cup} F'$.

THEOREM 3.21. *Suppose $p_g \geqslant 0$ and that C only has nodal singularities. Let G be the disconnected variety*

$$G = \operatorname{Pic}^0(C) \mathbin{\dot\cup} F \mathbin{\dot\cup} \operatorname{Pic}^1(C) \mathbin{\dot\cup} F',$$

where F' is a copy of F. There is a commutative algebraic group structure $+_G$ on G over k such that

1. *$G^0 = \operatorname{Pic}^0(C)$ with component group $G/G^0 \simeq \mathbb{Z}/4$;*
2. *F' is isomorphic to F as varieties via the inversion map -1_G;*
3. *the group law extends that on $H = \operatorname{Pic}(C)/\mathbb{Z}D_0 \simeq \operatorname{Pic}^0(C) \mathbin{\dot\cup} \operatorname{Pic}^1(C)$, where D_0 is the hyperelliptic class;*
4. *the group law defines a simply transitive action of H on $F \mathbin{\dot\cup} F'$ extending the following action of C^{sm}:*

$$\mathbb{P}X + (c) = -\tau(c)\mathbb{P}X, \quad -\mathbb{P}X + (c) = \tau(\overline{c})\mathbb{P}X,$$

with respect to which $x +_G x'$, for $x, x' \in F \mathbin{\dot\cup} F'$, is the unique divisor class sending $-x$ to x'.

The following result is immediate from Theorem 3.21 and Theorem 3.19.

COROLLARY 3.22. *Suppose $p_g \geqslant 0$ and that C only has nodal singularities. Then the short exact sequence*

$$1 \to S \to J(C) \to J(\widetilde{C}) \to 1$$

extends to a short exact sequence

$$1 \to S \to G \to \widetilde{G} \to 1,$$

where $G = \operatorname{Pic}^0(C) \mathbin{\dot\cup} F \mathbin{\dot\cup} \operatorname{Pic}^1(C) \mathbin{\dot\cup} F'$ and $\widetilde{G} = \operatorname{Pic}^0(\widetilde{C}) \mathbin{\dot\cup} \widetilde{F} \mathbin{\dot\cup} \operatorname{Pic}^1(\widetilde{C}) \mathbin{\dot\cup} \widetilde{F}'$ are the corresponding disconnected groups of four components.

Now, over the algebraic closure, after identifying F with $J(C)$, one can obtain a compactification of $J(C)$ by taking F_0. Recall that for any singular $[v] \in B$, we have the reduction map $\delta_v \colon F_0 \to \overline{F}_0$. Note that this map might not be a morphism of varieties. The composition of all the reduction maps gives $\delta \colon F_0 \to \widetilde{F} \simeq J(\widetilde{C})$. Every fiber of δ_v intersects $F_0 \backslash F$ at one point, obtained by taking the pre-image of $\mathbb{P}X \in \overline{F}_0$ under the map $v^\perp \to v^\perp/\langle v \rangle$.

COROLLARY 3.23. *Suppose $p_g \geqslant 0$ and that C has only nodal singularities. Then F_0 is a compactification of $J(C)$ by adding one point to each $\mathbb{G}_m$-factor of the fiber over $J(\widetilde{C})$.*

We expect that the condition of C having only nodal singularities is unnecessary. If Theorem 3.21 is proved without this condition, then Corollary 3.23 also holds without this condition. The compactification F_0 is not smooth.

Since multiplication by 2 and 4 are still surjective on J, we can lift F to a torsor of $J[4]$ by taking

$$F[4] = \{\mathbb{P}X \in F \mid \mathbb{P}X +_G \mathbb{P}X +_G \mathbb{P}X +_G \mathbb{P}X = 0\}.$$

When $\mathrm{Pic}^1(C)(k) \neq \emptyset$, the class $[F]$ is 2-torsion. For each $[D_1] \in \mathrm{Pic}^1(C)(k)$, we obtain a lift of F to a torsor of $J[2]$ by taking

$$F[2]_{[D_1]} = \{\mathbb{P}X \in F \mid \mathbb{P}X +_G \mathbb{P}X = [D_1]\}.$$

For the remainder of this section, we study the special case when $[D_1] = (c)$ comes from a point on the curve. The situation when c is a Weierstrass point is very similar to the generic case. The situation when c is not a Weierstrass point has much interesting geometry which we devote most of what follows to study.

3.2.1 *Example: Smooth rational Weierstrass point*

EXAMPLE 3.24. Suppose that C has a smooth rational Weierstrass point. By moving this point to ∞, we assume that Q_1 is singular with cone point $[v_\infty]$. Let $H = v_\infty^{\perp_{Q_2}}$ be the hyperplane in U orthogonal to v_∞ with respect to Q_2. Then $\tau(\infty)$ is induced by the linear map on U that fixes H and sends v_∞ to $-v_\infty$. Hence

$$F[2]_\infty = \{\mathbb{P}X \in F \mid \mathbb{P}X \subset B \cap \mathbb{P}H\}. \tag{3.5}$$

Just as in the generic case, when restricted to the $(2n+1)$-dimensional vector space H, the quadrics Q_1 and Q_2 span a regular pencil $\mathcal{L}_H$. Moreover, $Q_{1|H}$ is non-degenerate and $T_{|H}$ restricts to the self-adjoint operator on H associated to the pencil $\mathcal{L}_H$, as defined in (2.1). The right-hand side of (3.5) is precisely $L_{\{0,0,\dots,0\}}^{f,T_{|H}}$ as defined in the odd-dimension case. Now, $J[2]$ acts on $F[2]_\infty$ via the action of J and on $L_{\{0,0,\dots,0\}}^{f,T_{|H}}$ via the identification $J[2] \simeq \mathrm{Stab}_{\mathrm{PO}(H,Q_{1|H})}(T)$. As in the generic case, these two actions coincide.

3.2.2 *Example: Rational non-Weierstrass point*

EXAMPLE 3.25. Suppose that C has a rational non-Weierstrass point or, equivalently, that $\mathcal{L}$ has a rational quadric with discriminant 1. By moving this point to infinity, we assume that Q_1 has discriminant 1. The two rulings of Q_1 are then defined over k. Let Y_0 denote one of the rulings, and let $\infty \in C(k)$ denote the point corresponding to the quadric Q_1 and the ruling Y_0. Denote by ∞' the conjugate of ∞ under the hyperelliptic involution. As in the generic case, we have

$$F[2]_\infty = \{\mathbb{P}X \in F \mid \mathbb{P}X = \tau(\infty)\mathbb{P}X\} = \{\mathbb{P}X \in F \mid \mathrm{Span}\{\mathbb{P}X, \mathbb{P}(TX)\} \sim Y_0\}.$$

The latter condition means that $\mathrm{Span}\{\mathbb{P}X, \mathbb{P}(TX)\} \simeq \mathbb{P}^n$ is contained in Q_1 in the ruling Y_0.

Fix the monic polynomial $f(x)$ of degree $2n+2$ splitting completely over k^s and the quadratic form $Q = Q_1$ of discriminant 1. For every field k' containing k, define

$$V_f(k') = \{T\colon U \otimes k' \to U \otimes k' \mid T \text{ is self-adjoint and regular with minimal polynomial } f(x)\}.$$

For every field k' containing k and every $T \in V_f(k')$, let $W_T(k')$ denote the set of (linear) n-dimensional k'-subspaces X of $U \otimes k'$ such that $\mathrm{Span}\{X, TX\} \sim Y_0$. That is, the linear space $\mathrm{Span}\{X, TX\}$ is an $(n+1)$-dimensional isotropic subspace with respect to Q that lies inside the ruling Y_0. As before, we define

$$W_f(k') = \{(T, X) \mid T \in V_f(k'),\ X \in W_T(k')\}.$$

There is a Galois-equivariant action of $\mathrm{PSO}(U,Q) = \mathrm{SO}(U,Q)/(\pm 1)$ on W_f:

$$g.(T,X) = \left(gTg^{-1}, gX\right).$$

Recall that $U \otimes k^s$ breaks up as the orthogonal direct sum of generalized eigenspaces $U_{i,T}$ of T of dimension m_i. For any linear space X, we defined $\dim_{i,T}(X)$ to be the dimension of the maximal T-stable subspace in $(X \otimes k^s) \cap U_{i,T}$. For any sequence of integers $d_1, \ldots, d_{r+1}$ such that $0 \leqslant d_i \leqslant m_i/2$, we defined for any field k' containing k,

$$L_{\{d_1,\ldots,d_{r+1}\}}^{f,T}(k') = \{X \in W_T(k') \mid \dim_{i,T}(X) = d_i\}.$$

In view of the definition of F, we also define,

$$L_{\{d_1,\ldots,d_{r+1}\}}'^{f,T}(k') = \{X \in W_T(k') \mid \dim_{i,T}(\mathrm{Span}\{X, TX\}) = d_i\},$$
$$W_{\{d_1,\ldots,d_{r+1}\}}'^{f}(k') = \{(T,X) \mid T \in V_f(k'), \ X \in L_{\{d_1,\ldots,d_{r+1}\}}'^{f,T}(k')\}.$$

Note that when all the d_i are 0, we have $L_{\{0,\ldots,0\}}'^{f,T}(k') = F[2]_\infty(k')$.

We make no assumption on the reducibility of the associated hyperelliptic curve C but assume instead a weaker condition,

$$d_1 + \cdots + d_{r+1} < n+1 = \dim \mathrm{Span}\{X, TX\}. \tag{3.6}$$

Then condition (3.6) holds if and only if $\mathrm{Span}\{X, TX\}$ is not T-stable. Let s_1 denote the number of roots of f with odd multiplicity. Then the maximum $d_1 + \cdots + d_{r+1}$ could reach is $n+1 - s_1/2$. If (3.6) fails, then we must have $s_1 = 0$ and hence C is reducible. If one uses $L^{f,T}$ instead of $L'^{f,T}$ or if one does not assume (3.6), then there will be infinitely many choices for X when C is reducible. See Examples 3.37 and 3.38.

As one would expect from the odd case, the main theorems we are heading toward are the following.

THEOREM 3.26. *Suppose $d_1 + \cdots + d_{r+1} < n+1$. Let a denote the number of d_i equal to $m_i/2$. Then*

$$\left| L_{\{d_1,\ldots,d_{r+1}\}}'^{f,T}(k^s) \right| = 2^r/2^a.$$

THEOREM 3.27. *Suppose $d_1 + \cdots + d_{r+1} < n+1$. Then $\mathrm{PSO}(U,Q)(k^s)$ acts on $W_{\{d_1,\ldots,d_{r+1}\}}'^{f}(k^s)$ simply transitively if $a = 0$ and transitively if $a > 0$.*

We begin by studying the conjugation action of $\mathrm{PSO}(U,Q)$ on V_f. The proof of the following result is identical to that of Proposition 3.4.

PROPOSITION 3.28. *The action of $\mathrm{PSO}(U,Q)$ on V_f has a unique geometric orbit. For any $T \in V_f(k')$ defined over some field k' over k, its stabilizer scheme $\mathrm{Stab}(T)$ is isomorphic to $(\mathrm{Res}_{L'/k'}\,\mu_2)_{N=1}/\mu_2 \simeq J[2]$ as group schemes over k', where $L' = k'[x]/f(x)$. In particular, $\mathrm{Stab}_{\mathrm{PSO}(U,Q)}(T)(k^s)$ is an elementary abelian 2-group of order 2^r.*

Remark 3.29. A more explicit description for the stabilizer in terms of polynomials in T is almost identical to the odd case as given in Remark 3.5. For each $i = 1, \ldots, r+1$, define

$h_i^T(x) = f(x)/(x - \alpha_i)^{m_i}$. Then

$$\mu_2(k^s[T]^\times) = \left\{ \prod_{i \in I} \left(1 - 2\frac{h_i^T(T)}{h_i^T(\alpha_i)} \right) \right\}_{I \subset \{1,\ldots,r+1\}, 2 \mid |I|}$$

$$= \left\{ 1 - 2 \sum_{i \in I} \frac{h_i^T(T)}{h_i^T(\alpha_i)} \right\}_{I \subset \{1,\ldots,r+1\}, 2 \mid |I|} .$$

For any $T \in V_f(k^s)$, its stabilizer J_T in $\mathrm{PSO}(U,Q)(k^s)$ acts on $L'^{f,T}_{\{d_1,\ldots,d_{r+1}\}}(k^s)$. Then just as Theorem 3.2 follows from Theorems 3.1 and 3.6, Theorem 3.27 follows from Theorems 3.26 and 3.30.

THEOREM 3.30. *Suppose $d_1 + \cdots + d_{r+1} < n + 1$. Let a denote the number of d_i equal to $m_i/2$. Then for any $X \in L'^{f,T}_{\{d_1,\ldots,d_{r+1}\}}(k^s)$,*

$$|\operatorname{Stab}_{J_T}(X)| = 2^a .$$

We will prove Theorems 3.26 and 3.30 via the same series of reductions as in the odd-dimension case.

In what follows, it turns out that one should forget about the rulings in the following reductions. Namely, consider instead

$$W_T^*(k^s) = \{X \in \mathrm{Gr}(n, U \otimes k^s) \mid \mathrm{Span}\{X, TX\} \text{ is } (n+1)\text{-dimensional and isotropic}\} .$$

Observe that $W_T^*(k^s)$ can be divided into two components, one of which is $W_T(k^s)$, corresponding to which ruling $\mathrm{Span}\{X, TX\}$ lies in. The two components are in bijection with each other via an element g_0 in $\mathrm{Stab}_{\mathrm{PO}}(T)(k^s)$ that does not lie in $\mathrm{Stab}_{\mathrm{PSO}}(T)(k^s)$. One similarly defines $L^{f,T,*}_{\{d_1,\ldots,d_{r+1}\}}(k^s)$ and $L'^{f,T,*}_{\{d_1,\ldots,d_{r+1}\}}(k^s)$.

Once again, most of the following results are proved by hardcore linear algebra. We recommend the interested readers to work out the proofs themselves.

Reduction on $d_1, \ldots, d_{r+1}$. Suppose $X \in L'^{f,T,*}_{\{d_1,\ldots,d_{r+1}\}}(k^s)$ with $d_i \geqslant 1$. Let v_i denote an eigenvector of T corresponding to α_i. Since T is regular, v_i is unique up to scaling. The assumption $d_i \geqslant 1$ implies $v_i \in \mathrm{Span}\{X, TX\}$. Hence

$$X \subset \mathrm{Span}\{X, TX\}^\perp \subset v_i^\perp .$$

Let b denote the bilinear form associated to Q. As before, the data (Q, T) descends to $(\overline{Q}_i, \overline{T}_i)$ on $\overline{U}_i := v_i^\perp / \langle v_i \rangle$ and $\overline{T}_i$ is regular with characteristic polynomial $f(x)/(x - \alpha_i)^2$. Let $\overline{X}_i$ denote the image of X in $\overline{U}_i$. Then $\mathrm{Span}\{\overline{X}_i, \overline{T}_i \overline{X}_i\}$ is an isotropic n-plane with respect to $\overline{Q}_i$, and the dimensions of the maximal $\overline{T}_i$-stable subspaces in the intersection of $\mathrm{Span}\{\overline{X}_i, \overline{T}_i \overline{X}_i\}$ with the generalized eigenspaces are $d_1, \ldots, d_i - 1, \ldots, d_{r+1}$. Suppose, for a contradiction, $v_i \notin X$. Then $\mathrm{Span}\{X, v_i\}$ is an isotropic $(n+1)$-plane. Hence $\mathrm{Span}\{X, TX\}$ is either $\mathrm{Span}\{X, v_i\}$ or $g_0 \mathrm{Span}\{X, v_i\}$, both of which are T-stable, violating condition (3.6). Hence $v_i \in X$ and so $\overline{X}_i$ is $(n-1)$-dimensional. We denote this reduction step by

$$L'^{f,T,*}_{\{d_1,\ldots,d_{r+1}\}}(k^s) \xrightarrow[\delta_i]{\sim} L'^{f/(x-\alpha_i)^2, \overline{T}_i,*}_{\{d_1,\ldots,d_i-1,\ldots,d_{r+1}\}}(k^s) .$$

The map δ_i is bijective, its inverse is given by taking the pre-image of the projection map $H_i \to \overline{U}_i$.

The stabilizers are affected in the same manner as in the odd case. We summarize this reduction step in the following proposition.

PROPOSITION 3.31. *Suppose $d_i \geqslant 1$; then there is a bijection*

$$\delta_i \colon L'^{f,T,*}_{\{d_1,\dots,d_{r+1}\}}(k^s) \overset{\sim}{\longrightarrow} L'^{f/(x-\alpha_i)^2,\overline{T}_i,*}_{\{d_1,\dots,d_i-1,\dots,d_{r+1}\}}(k^s)\,.$$

When $m_i \geqslant 3$, the map δ_i does not change the sizes of the stabilizers. When $m_i = 2$, the map δ_i lowers the sizes of the stabilizers by a factor of 2.

The map δ_i can be described projectively as intersecting the quadric defined by Q with the tangent plane to v_i, then projecting away from v_i. Such an operation does not preserve the rulings. Two (projective) n-planes in Q lying in the same ruling could be sent to different rulings via this procedure. For example, take a smooth quadric in $\mathbb{P}^7$ and two 3-planes Y_1 and Y_2 on the quadric intersecting at a line. Then these two 3-planes lie on the same ruling. If the tangent plane to v contains this line, then the images of Y_1 and Y_2 lie in different rulings since their intersection codimension is 1. If the tangent plane to v_i meets this line at a point, then the images of Y_1 and Y_2 lie in the same ruling as their intersection codimension is 2. Similar examples can be written down when Y_1 and Y_2 lie on different rulings. This is the main reason why forgetting the rulings is more convenient.

Reduction on f. By the above reduction step, it remains to study $L'^{f,T,*}_{\{0,0,\dots,0\}}(k^s)$. Let $X \in W_T^*(k^s)$ be arbitrary. By Lemma 3.13, the following conditions are equivalent:

1. The linear space $\mathrm{Span}\{X, TX\}$ contains no T-stable subspace.

2. We have $v \notin X$ for any eigenvector v whose eigenvalue has multiplicity at least 2.

3. We have $X \not\subseteq v^\perp$ for any eigenvector v whose eigenvalue has multiplicity at least 2.

The arguments in the odd-dimension case rely much on the second and the third conditions. Since they are equivalent to the first condition, we see that these arguments carry through immediately and will not repeat them. The main difference to the odd-dimension case is due to dimension reasons which one handles by replacing certain X (and $\overline{X}$) by $\mathrm{Span}\{X, TX\}$ (and $\mathrm{Span}\{\overline{X}, \overline{TX}\}$).

Suppose that α is a root of f of multiplicity $m \geqslant 2$. Let $X \in L'^{f,T,*}_{\{0,0,\dots,0\}}(k^s)$ be arbitrary. Let v denote an eigenvector of T with eigenvalue α. Then, as before, the non-degenerate quadratic form Q descends to a non-degenerate quadratic form $\overline{Q}$ on $\overline{U} = v^\perp/\langle v \rangle$ and T descends to a regular self-adjoint operator $\overline{T}$ on $\overline{U}$ with characteristic polynomial $f(x)/(x-\alpha)^2$. Let $\overline{X}$ denote the image of $X \cap v^\perp$ under the quotient map $U \to U/\langle v \rangle$.

PROPOSITION 3.32. *The above map sending X to $\overline{X}$ defines a surjection*

$$L'^{f,T,*}_{\{0,0,\dots,0\}}(k^s) \longrightarrow L'^{f/(x-\alpha)^2,\overline{T},*}_{\{0,0,\dots,0\}}(k^s)\,.$$

This map is bijective if $m > 2$ and is two-to-one if $m = 2$. In both cases,

$$\left| \mathrm{Stab}_{J_T}(X) \right| = \left| \mathrm{Stab}_{J_{\overline{T}}}(\overline{X}) \right| \quad \text{for any } X \in L'^{f,T,*}_{\{0,\dots,0\}}(k^s)\,.$$

Proof. Take any $\overline{X} \in L'^{f/(x-\alpha)^2,\overline{T},*}_{\{0,0,\dots,0\}}(k^s)$. As in the odd-dimension case, let b_α denote the bilinear form

$$b_\alpha(u, u') = b(u, (T-\alpha)u')\,,$$

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where b denotes the bilinear form associated to Q. Then b_α descends to a non-degenerate bilinear form on the $(2n+1)$-dimensional vector space $U/\langle v \rangle$. Denote by $\perp_\alpha$ the perpendicular space with respect to b_α. We claim that b_α descends to a non-degenerate bilinear form on the 2-dimensional vector space $\overline{Y} = \mathrm{Span}\{\overline{X}, \overline{TX}\}^{\perp_\alpha}/\overline{X}$. Indeed, we know a priori that b_α descends to a non-degenerate form on the 3-dimensional space $\overline{X}^{\perp_\alpha}/\overline{X}$. We claim

$$\overline{X}^{\perp_\alpha} = \mathrm{Span}\left\{ \mathrm{Span}\left\{\overline{X}, \overline{TX}\right\}^{\perp_\alpha}, \overline{TX} \right\}.$$

Indeed, since $\mathrm{Span}\{\overline{X}, \overline{TX}\}$ is isotropic with respect to b, we have $\overline{TX} \subset \overline{X}^{\perp_\alpha}$. It remains to show $\overline{TX} \nsubseteq \mathrm{Span}\{\overline{X}, \overline{TX}\}^{\perp_\alpha}$. In other words, we need to show that $\mathrm{Span}\{\overline{X}, \overline{TX}\}$ is not isotropic with respect to b_α. Suppose, for a contradiction, that $\mathrm{Span}\{\overline{X}, \overline{TX}\}$ is isotropic with respect to b_α. Then inside $\overline{U}$,

$$\overline{T}^2\overline{X} \subset \mathrm{Span}\left\{\overline{X}, \overline{TX}\right\}^{\perp} = \mathrm{Span}\left\{\overline{X}, \overline{TX}\right\}.$$

Hence the entire $\mathrm{Span}\{\overline{X}, \overline{TX}\}$ is $\overline{T}$-stable. This gives a contradiction.

Now, take any $w \in \mathrm{Span}\{\overline{X}, \overline{TX}\}^{\perp_\alpha}$. One can first find some $w' \in \overline{X}^{\perp_\alpha}$ such that $b_\alpha(w, w')$ is non-zero. Let u' be an element of $\overline{TX}$ such that $w' - u' \in \mathrm{Span}\{\overline{X}, \overline{TX}\}^{\perp_\alpha}$. Then $b_\alpha(w, w' - u') = b_\alpha(w, w') \neq 0$. Hence we have shown that b_α is non-degenerate on the 2-dimensional space $\overline{Y}$. There are then two 1-dimensional isotropic lines. Denote by $\overline{X}_1$ and $\overline{X}_2$ their pre-images in $\mathrm{Span}\{\overline{X}, \overline{TX}\}^{\perp_\alpha}$.

The case $m \geqslant 3$ is identical to the odd-dimension case: one of $\overline{X}_1$ and $\overline{X}_2$ does not lift while the other one lifts uniquely to some X^w. The condition that $\overline{X}_1$ and $\overline{X}_2$ lie inside $\mathrm{Span}\{\overline{X}, \overline{TX}\}^{\perp_\alpha}$ ensures that $\mathrm{Span}\{X^w, TX^w\}$ is isotropic with respect to Q.

Now, suppose $m = 2$. Write $\overline{X}_1 = \mathrm{Span}\{w_1 + \langle v \rangle, \overline{X}\}$ and $\overline{X}_2 = \mathrm{Span}\{w_2 + \langle v \rangle, \overline{X}\}$. The only difference to the odd-dimension case is the proof of $w_1 \notin v^\perp$. Suppose, for a contradiction, $w_1 \in v^\perp$. Since $\mathrm{Span}\{\overline{X}, \overline{TX}\}$ is not isotropic with respect to b_α, we see that we have $\mathrm{Span}\{\overline{X}, \overline{TX}\} \neq \mathrm{Span}\{\overline{X}, w_1 + \langle v \rangle\}$ and so $\mathrm{Span}\{\overline{X}, \overline{TX}, w + \langle v \rangle\}$ is an $(n+1)$-dimensional subspace of $v^\perp/\langle v \rangle$. As in the odd case, b_α is non-degenerate on $v^\perp/\langle v \rangle$ because $T - \alpha$ acts invertibly on $v^\perp/\langle v \rangle$. However, taking $\perp_\alpha$ inside $v^\perp/\langle v \rangle$, we see that we have

$$\mathrm{Span}\left\{\overline{X}, \overline{TX}, w_1 + \langle v \rangle\right\}^{\perp_\alpha} \supset \mathrm{Span}\{w_1 + \langle v \rangle, \overline{X}\}.$$

The left-hand side has dimension $n - 1$ while the right-hand side has dimension n, giving a contradiction. The rest of the argument is identical to the odd-dimension case: $\overline{X}_1$ and $\overline{X}_2$ each has a unique lift.

Stabilizers behave in the same way as in the odd-dimension case. $\qquad\square$

PROPOSITION 3.33. *We have $|L'^{f,T,*}_{\{0,0,\dots,0\}}(k^s)| = 2^{r+1}$, and every element has trivial stabilizer in J_T.*

Proof. Apply the reduction steps like in the odd-dimension case. There are now five base cases, which we illustrate below as examples.

EXAMPLE 3.34 (Generic case). Suppose that reduction terminates with $f(x) = \prod_{i=1}^{r+1}(x - \alpha_i)$ with $r \geqslant 3$. In this case, one can apply the theory for the non-singular case discussed in Example 2.30 and obtain $|L'^{f,T,*}_{\{0,0,\dots,0\}}(k^s)| = 2^{r+1}$.

EXAMPLE 3.35. Suppose that reduction terminates with $f(x) = (x - \alpha)(x - \beta)(x - \gamma)^2$. If one tries to apply reduction again on γ, then $\overline{X}$ becomes 0-dimensional. Let u, v, w_1 denote eigenvectors

of T with eigenvalues α, β, γ, respectively, and let w_2 be such that $(T - \gamma)w_2 = w_1$. We seek coefficients $c_1, \ldots, c_4$ such that $X = \langle c_1 u + c_2 v + c_3 w_1 + c_4 w_2 \rangle$ lies in $L_{0,0,0}^{'f,T,*}(k^s)$. Set

$$\Omega_1 = b(u, u) \neq 0, \quad \Omega_2 = b(v, v) \neq 0, \quad \Gamma_3 = b(w_1, w_2) \neq 0, \quad \Gamma_4 = b(w_2, w_2).$$

Then the condition that $\mathrm{Span}\{X, TX\}$ is an isotropic 2-plane becomes

$$\begin{pmatrix} \Omega_1 & \Omega_2 & \Gamma_3 & \Gamma_4 \\ (\gamma - \alpha)\Omega_1 & (\gamma - \beta)\Omega_2 & 0 & \Gamma_3 \\ (\gamma - \alpha)^2\Omega_1 & (\gamma - \beta)^2\Omega_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1^2 \\ c_2^2 \\ 2c_3 c_4 \\ c_4^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since Γ_3, Ω_1, Ω_2 are non-zero, the above matrix has a 1-dimensional kernel. Moreover, if any one of c_1, c_2, c_4 is zero, then they are all zero and X is of the form $\langle c_3 w_1 \rangle$, which does not lie in $L_{0,0,0}^{'f,T,*}(k^s)$. Now, given non-zero c_1, c_2, c_4 obtained by taking a non-zero vector in the 1-dimensional kernel and taking square roots of the corresponding entries, there is a unique solution for c_3. Therefore, there are $8 = 2^3$ choices for X depending on which square roots one chooses for c_1, c_2, c_4.

EXAMPLE 3.36. Suppose that reduction terminates with $f(x) = (x - \alpha)^3(x - \beta)$. Let u_1 and v denote eigenvectors of T with eigenvalues α and β, respectively, and let u_2 and u_3 be such that $(T - \alpha)^2 u_3 = (T - \alpha)u_2 = u_1$. We seek coefficients $c_1, \ldots, c_4$ such that $X = \langle c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 v \rangle$ lies in $L_{0,0}^{'f,T,*}(k^s)$. Set

$$\Omega = b(v, v) \neq 0, \quad \Gamma_4 = b(u_1, u_3) = b(u_2, u_2) \neq 0, \quad \Gamma_5 = b(u_2, u_3), \quad \Gamma_6 = b(u_3, u_3).$$

Then the condition that $\mathrm{Span}\{X, TX\}$ is an isotropic 2-plane becomes

$$\begin{pmatrix} \Gamma_4 & \Gamma_5 & \Gamma_6 & \Omega \\ 0 & \Gamma_4 & \Gamma_5 & (\beta - \alpha)\Omega \\ 0 & 0 & \Gamma_4 & (\beta - \alpha)^2\Omega \end{pmatrix} \begin{pmatrix} c_2^2 + 2c_1 c_3 \\ 2c_2 c_3 \\ c_3^2 \\ c_4^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since Γ_4 and Ω are non-zero, the above matrix has a 1-dimensional kernel, and if any one of c_2, c_3, c_4 is zero, then all of them are zero and X is of the form $\langle c_1 u_1 \rangle$, which does not lie in $L_{0,0}^{'f,T,*}(k^s)$. Now, given non-zero c_3 and c_4 obtained by taking a non-zero vector in the 1-dimensional kernel and taking square roots of the corresponding entries, there is a unique solution for c_1 and c_2. Therefore, there are $4 = 2^2$ choices for X depending on which square roots one chooses for c_3 and c_4.

EXAMPLE 3.37. Suppose that reduction terminates with $f(x) = (x - \alpha)^2(x - \beta)^2$. Let u_1 and v_1 denote eigenvectors of T with eigenvalues α and β, respectively, and let u_2 and v_2 be such that $(T - \alpha)u_2 = u_1$ and $(T - \beta)v_2 = v_1$. We seek coefficients $c_1, \ldots, c_4$ such that $X = \langle c_1 u_1 + c_2 u_2 + c_3 v_1 + c_4 v_2 \rangle$ lies in $L_{0,0}^{'f,T,*}(k^s)$. Set

$$\Gamma_3 = b(u_1, u_2) \neq 0, \quad \Gamma_4 = b(u_2, u_2), \quad \Omega_3 = b(v_1, v_2) \neq 0, \quad \Omega_4 = b(v_2, v_2).$$

Then the condition that $\mathrm{Span}\{X, TX\}$ is an isotropic 2-plane becomes

$$\begin{pmatrix} \Gamma_3 & \Gamma_4 & \Omega_3 & \Omega_4 \\ 0 & \Gamma_3 & (\beta - \alpha)\Omega_3 & \Omega_3 + (\beta - \alpha)\Omega_4 \\ 0 & 0 & (\beta - \alpha)^2\Omega_3 & 2(\beta - \alpha)\Omega_3 + (\beta - \alpha)^2\Omega_4 \end{pmatrix} \begin{pmatrix} 2c_1 c_2 \\ c_2^2 \\ 2c_3 c_4 \\ c_4^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since Γ_3 and Ω_3 are non-zero, the above matrix has a 1-dimensional kernel. If any one of c_2 and c_4 is zero, then both of them are zero and X is of the form $\langle c_1 u_1 + c_3 v_1 \rangle$. In this case, $\mathrm{Span}\{X, TX\}$ contains either $\langle u_1 \rangle$ or $\langle v_1 \rangle$, both of which are T-stable, thereby forcing $X \notin L'^{f,T,*}_{0,0}(k^s)$. Note that if c_1 and c_3 are both non-zero, then X satisfies the weaker condition that X contains no non-zero T-stable subspace while $X \in L'^{f,T,*}_{1,1}(k^s)$ violates condition (3.6). It is clear that there are infinitely many such X.

Now, given non-zero c_2 and c_4 obtained by taking a non-zero vector in the 1-dimensional kernel and taking square roots of the corresponding entries, there is a unique solution for c_1 and c_3. Therefore, there are $4 = 2^2$ choices for X depending on which square roots one chooses for c_2 and c_4.

EXAMPLE 3.38. Suppose that reduction terminates with $f(x) = (x - \alpha)^4$. Let u_1 denote an eigenvector of T with eigenvalue α, and let u_2, u_3, u_4 be such that

$$(T - \alpha)^3 u_4 = (T - \alpha)^2 u_3 = (T - \alpha) u_2 = u_1 \,.$$

We seek coefficients $c_1, \ldots, c_4$ such that $X = \langle c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4 \rangle$ lies in $L^{f,T,*}_0(k^s)$. Set

$$\Gamma_5 = b(u_1, u_4) \neq 0 \,, \quad \Gamma_6 = b(u_2, u_4) = b(u_3, u_3) \,, \quad \Gamma_7 = b(u_3, u_4), \quad \Gamma_8 = b(u_4, u_4) \,.$$

Then the condition that $\mathrm{Span}\{X, TX\}$ is an isotropic 2-plane becomes

$$\begin{pmatrix} \Gamma_5 & \Gamma_6 & \Gamma_7 & \Gamma_8 \\ 0 & \Gamma_5 & \Gamma_6 & \Gamma_7 \\ 0 & 0 & \Gamma_5 & \Gamma_6 \end{pmatrix} \begin{pmatrix} 2c_1 c_4 + 2c_2 c_3 \\ 2c_2 c_4 + c_3^2 \\ 2c_3 c_4 \\ c_4^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \,.$$

Since Γ_5 is non-zero, the above matrix has a 1-dimensional kernel. If c_4 is zero, then c_3 is also zero. In this case, any X of the form $\langle c_1 u_1 + c_2 u_2 \rangle$ solves the above equation. However, for all such lines, $\mathrm{Span}\{X, TX\}$ contains the T-stable subspace $\langle u_1 \rangle$, thereby forcing $X \notin L'^{f,T,*}_0(k^s)$. Note that if $c_1 \neq 0$, then $\mathrm{Span}\{X, TX\} = \langle u_1, u_2 \rangle$ and $X \in L'^{f,T,*}_2(k^s)$, violating condition (3.6), while every such X still satisfies the weaker condition that it contains no non-zero T-stable subspace.

Now, given a non-zero c_4 obtained by taking a non-zero vector in the 1-dimensional kernel and taking a square root of the corresponding entry, there is a unique solution for c_1, c_2 and c_3. Therefore, there are $2 = 2^1$ choices for X depending on which square root one chooses for c_4.

We have now completed the proof of Proposition 3.33. $\qquad\square$

Proof of Theorems 3.26 and 3.30. Applying Proposition 3.31 repeatedly gives a bijection

$$L'^{f,T,*}_{\{d_1,\ldots,d_{r+1}\}}(k^s) \xrightarrow[\delta]{\sim} L'^{\prod_i (x-\alpha_i)^{m_i - 2d_i}, T', *}_{\{0,0,\ldots,0\}}(k^s) \,,$$

and for any $X \in L'^{f,T,*}_{\{d_1,\ldots,d_{r+1}\}}(k^s)$,

$$|\,\mathrm{Stab}_{J_T}(X)| = 2^a |\,\mathrm{Stab}_{J_{T'}}(\delta(X))| \,.$$

This proves Theorem 3.30. The polynomial $g(x) = \prod_i (x - \alpha_i)^{m_i - 2d_i}$ has $r + 1 - a$ distinct roots, hence applying Proposition 3.33 to g and then dividing by 2 to go from $|L'^{f,T,*}|$ to $|L'^{f,T}|$ completes the proof of Theorem 3.26. $\qquad\square$

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