Week 1: Mock Putnam 1

1: Let n be a positive integer and let p be a prime such that $p^p \mid n!$. Prove that $p^{p+1} \mid n!$.

2: Prove that every integer is a sum of 5 cubes. (It is an open problem whether every integer is a sum of 4 cubes.)

3: Let p be a prime. Prove that there does not exist positive integers $a_1, a_2, \ldots, a_p$, not all equal, such that for any positive integer n, the number $(a_1+n)(a_2+n)\cdots(a_p+n)$ is a perfect power. Here, a perfect power is a positive integer of the form a^b for some $a, b \in \mathbb{Z}$ with $a, b \geq 2$.

4: Find all integers a, b, c, d such that $1 < a < b < c < d$ and

$$(a-1)(b-1)(c-1)(d-1) \mid abcd - 1.$$

5: Let $a \neq -1/2$ be a real number. Find all continuous functions $f : [0,1] \to \mathbb{R}$ such that for any $x \in [0,1]$,

$$f(x^2) + 2af(x) = (x+a)^2.$$

6: Let S be a finite set of positive integers such that

$$\sum_{a \in S} \arctan \frac{1}{a} < \frac{\pi}{2}.$$

Prove that there exists a finite set T of positive integers containing S such that

$$\sum_{a \in T} \arctan \frac{1}{a} = \frac{\pi}{2}.$$

Week 1: Sketch of proofs

1: Looking at p-adic valuation works.

If $n < p^2$, then at most $p-1$ positive integers up to n are divisible by p, and they are all not divisible by p^2. This would imply $\nu_p(n!) \le p-1$; a contradiction.

If $n \ge p^2$, then p^{p+1} divides $p \cdot (2p) \cdot \ldots \cdot (p^2)$, which divides $n!$.

Note: Legendre's formula allows for another solution. This will be left as an exercise for the reader.

2: The major formula is $(x+1)^3 + (x-1)^3 + (-x)^3 + (-x)^3 = 6x$.

Indeed, $(x+1)^3 = x^3+3x^2+3x+1$ and $(x-1)^3 = x^3-3x^2+3x-1$, so $(x+1)^3+(x-1)^3 = 2x^3+6x$. This proves the formula.

Due to the formula, $6x$, $6x+1$, $6x+8$, $6x+27$, $6x-8$, and $6x-1$ are all sums of 5 cubes. On the other hand, $0, 1, 8, 27, -8, -1$ forms a complete residue classes mod 6. That is, every integer is congruent to one of these six number mod 6. Thus, every number is either of form $6x$, $6x+1$, $6x+8$, $6x+27$, $6x-8$, or $6x-1$ for some integer x. This proves that every number is a sum of 6 cubes.

Note: By working mod 9, it is easy to prove that 4 is not a sum of 3 cubes.)

3: Let $b_1, b_2, \ldots, b_k$ be the distinct integers among $a_1, a_2, \ldots, a_p$, and for each i, let m_i be the multiplicity of b_i in the a_js. Then we have

$$(a_1+n)(a_2+n)\cdots(a_p+n) = (b_1+n)^{m_1}(b_2+n)^{m_2}\cdots(b_k+n)^{m_k}.$$

Choose arbitrary distinct primes $p_1, p_2, \ldots, p_k$, all greater than the b_is. By Chinese Remainder Theorem, there exists a positive integer n such that

$$n \equiv p_i - b_i \pmod{p_i^2} \implies \nu_{p_i}(b_i+n) = 1.$$

Furthermore, for each $i \ne j$, we see that $p_i > |b_i - b_j| > 0$, and so $n + b_j = (n+b_i) + (b_j - b_i)$ is not divisible by p_i. As a result, for each i,

$$\nu_{p_i}\left((b_1+n)^{m_1}(b_2+n)^{m_2}\cdots(b_k+n)^{m_k}\right) = m_i.$$

If $(b_1+n)^{m_1}(b_2+n)^{m_2}\cdots(b_k+n)^{m_k}$ is a prime power, then there exists $m > 1$ such that $m \mid m_i$ for each i. But then m divides $m_1 + m_2 + \ldots + m_k = p$. This implies $m = p$, and thus $k = 1$; a contradiction.

4: **Answer.** $(2, 4, 10, 80)$ and $(3, 5, 17, 255)$.

We need to combine bound argument with divisibility argument.

Write $k(a-1)(b-1)(c-1)(d-1) = abcd - 1$ for some (positive) integer k. Note that $abc(d-1) < abcd - 1 < abcd$; the first inequality holds since $abc > 1$. Thus, we get the bound

$$\frac{abc}{(a-1)(b-1)(c-1)} < k < \frac{abcd}{(a-1)(b-1)(c-1)(d-1)}. \tag{4.1}$$

By the given equality, $k(a-1)(b-1)(c-1)(d-1)$ and $abcd$ are coprime. Thus, each of the five integers $k, a-1, b-1, c-1, d-1$ are coprime with the four integers a, b, c, d. We now have two cases based on the parity of $abcd-1$.

- Case 1: $abcd-1$ is odd.

  Then $k(a-1)(b-1)(c-1)(d-1)$ is odd. In particular, k is odd and a, b, c, d are even. The latter yields $a \geq 2$, $b \geq 4$, $c \geq 6$, and $d \geq 8$. By (4.1),

$$k \leq \frac{2 \cdot 4 \cdot 6 \cdot 8}{1 \cdot 3 \cdot 5 \cdot 7} = \frac{384}{105} < 4.$$

  Since k is odd and $k \geq 2$, this implies $k = 3$.

  If $a \geq 4$, then $b \geq 6$, $c \geq 8$, $d \geq 10$, and

$$\frac{abcd}{(a-1)(b-1)(c-1)(d-1)} = \frac{4 \cdot 6 \cdot 8 \cdot 10}{3 \cdot 5 \cdot 7 \cdot 9} = \frac{1920}{945} < 3.$$

  This contradicts (4.1), so $a = 2$.

  Now suppose that $b \geq 6$. Since $k = 3$ has to be coprime to b, we get $b \geq 8$. Thus $c \geq 10$, $d \geq 12$, and

$$\frac{abcd}{(a-1)(b-1)(c-1)(d-1)} \leq \frac{2 \cdot 8 \cdot 10 \cdot 12}{1 \cdot 7 \cdot 9 \cdot 11} = \frac{1920}{693} < 3.$$

  This contradicts (4.1), so $b = 4$.

  The equality $k(a-1)(b-1)(c-1)(d-1) = abcd - 1$ now becomes

$$9(c-1)(d-1) = 8cd - 1 \iff cd - 9(c+d) + 10 = 0 \iff (c-9)(d-9) = 71.$$

  Since $c-9 < d-9$ and 71 is prime, this yields either $(c-9, d-9) = (-71, -1)$ or $(c-9, d-9) = (1, 71)$. The former does not work since then $c = -62 < 0$. The latter yields $c = 10$, $d = 80$, and so $(a, b, c, d) = (2, 4, 10, 80)$.

- Case 2: $abcd-1$ is even.

  Then a, b, c, d are odd, so $a \geq 3$, $b \geq 5$, $c \geq 7$, $d \geq 9$, and

$$k \leq \frac{3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8} = \frac{945}{384} < 3.$$

  Since $k \geq 2$, this implies $k = 2$.

  If $a \geq 5$, then $b \geq 7$, $c \geq 9$, $d \geq 11$, and

$$\frac{abcd}{(a-1)(b-1)(c-1)(d-1)} \leq \frac{5 \cdot 7 \cdot 9 \cdot 11}{4 \cdot 6 \cdot 8 \cdot 10} = \frac{3465}{1920} < 2.$$

  This contradicts (4.1), so $a = 3$.

  If $b \geq 7$, then $b \geq 9$ since $b - 1 = 6$ is not coprime with $a = 3$. Then $c \geq 11$ and $d \geq 13$, but $13 - 1$ is not coprime with 3, so $d \geq 15$ and

$$\frac{abcd}{(a-1)(b-1)(c-1)(d-1)} \leq \frac{3 \cdot 9 \cdot 11 \cdot 15}{2 \cdot 8 \cdot 10 \cdot 14} = \frac{4455}{2240} < 2.$$

This contradicts (4.1), so $b = 5$.

Now the equality $k(a-1)(b-1)(c-1)(d-1) = abcd - 1$ becomes

$$16(c-1)(d-1) = 15cd - 1 \iff cd - 16(c+d) + 17 = 0 \iff (c-16)(d-16) = 239.$$

Note that 239 is also prime. Since $c-16 < d-16$, this yields either $(c-16, d-16) = (-239, -1)$ or $(c-16, d-16) = (1, 239)$. The former does not work since $c = -223 < 0$. The latter yields $c = 17$, $d = 255$, and so $(a, b, c, d) = (3, 5, 17, 255)$.

5: Answer. $f(x) = x + \dfrac{a^2}{2a+1}$.

Plugging $x = 0$ yields $(2a+1)f(0) = a^2$, while plugging $x = 1$ yields $(2a+1)f(1) = (a+1)^2$. For convenience, let $C = \dfrac{a^2}{2a+1}$; then $f(0) = C$ and $f(1) = \dfrac{(a+1)^2}{2a+1} = C + 1$. At this point, it is natural to guess that $f(x) = x + \dfrac{a^2}{2a+1}$ works.

Consider the function $g(x) = f(x) - x - C$. Substituting $f(x) = g(x) + x + C$ yields

$$\begin{aligned} f(x^2) + 2af(x) &= g(x^2) + x^2 + C + 2a(g(x) + x + C) \\ &= g(x^2) + 2ag(x) + x^2 + 2ax + (2a+1)C \\ &= g(x^2) + 2ag(x) + (x+a)^2. \end{aligned}$$

Thus, $f(x^2) + 2af(x) = (x+a)^2$ for all $x \in [0, 1]$ if and only if g satisfies the functional equation

$$g(x^2) + 2ag(x) = 0 \quad \forall x \in [0, 1].$$

Clearly, $g \equiv 0$ works, so $f(x) = x + C$ satisfies the original functional equation. It remains to show that the only g satisfying the above functional equation is $g \equiv 0$.

Rewrite the above functional equation as $g(x^2) = -2ag(x)$. Plugging $x = 0$ and $x = 1$ yields $g(0) = g(1) = 0$. If $a = 0$, then $g(x^2) = 0$ for all $x \in [0, 1]$ and we are done, so now assume that $a \neq 0$.

By induction on k, we get $g(x^{2^k}) = (-2a)^k g(x)$ for any $x \in [0, 1]$ and non-negative integer k. For each $x \in (0, 1)$, the sequence $(x^{2^k})_{k \geq 0}$ converges to 0. By continuity,

$$\lim_{k \to \infty} (-2a)^k g(x) = \lim_{k \to \infty} g\left(x^{2^k}\right) = g(0) = 0.$$

This forces $g(x) = 0$ for each $x \in (0, 1)$ if either $|-2a| > 1$ or $-2a = -1$ holds. It remains to consider the case $0 < |-2a| < 1$.

The formula $g(x^{2^k}) = (-2a)^k g(x)$ yields $g(x^{1/2^k}) = (-2a)^{-k} g(x)$ for all $k \geq 0$ and $x \in [0, 1]$. For each $x \in (0, 1)$, the sequence $(x^{1/2^k})_{k \geq 0}$ converges to 1. By continuity,

$$\lim_{k \to \infty} (-2a)^{-k} g(x) = \lim_{k \to infty} g\left(x^{1/2^k}\right) = g(1) = 0.$$

Since $|-2a| < 1$, this gives $g(x) = 0$ for all $x \in (0, 1)$. Together with $g(0) = g(1) = 0$, we get $g \equiv 0$.

6: The greedy algorithm suggests constructing $t_1, t_2, \ldots$ such that t_{n+1} is the smallest integer larger than everything in S and all of $t_1, \ldots, t_n$ and that adding $\arctan(1/t_{n+1})$ does not exceed $\pi/2$. Suppose for a contradiction that the sequence (t_n) continues forever. Let

$$b_n = \tan\left(\frac{\pi}{2} - \sum_{a \in S} \arctan\frac{1}{a} - \sum_{i=1}^{n} \arctan\frac{1}{t_i}\right).$$

Then (b_n) is a sequence of positive rational numbers approaching 0 with

$$\frac{1}{b_{n+1}} = \frac{\frac{1}{b_n} + \frac{1}{t_{n+1}}}{1 - \frac{1}{b_n t_{n+1}}}, \qquad \text{that is} \qquad b_{n+1} = \frac{b_n t_{n+1} - 1}{b_n + t_{n+1}}.$$

Our greedy algorithm requires $t_{n+1} \geq \lceil \frac{1}{b_n} \rceil$. What if we define $t_{n+1} = \lceil \frac{1}{b_n} \rceil$ for all n? Suppose we write $b_n = p_n/q_n$ in reduced form. Then

$$\frac{p_{n+1}}{q_{n+1}} = \frac{p_n t_{n+1} - q_n}{p_n + t_{n+1} q_n}.$$

Note that $t_{n+1} < 1 + q_n/p_n$ and so $p_n t_{n+1} - q_n < p_n$. So we get $p_{n+1} < p_n$, which can't continue forever.

The problem with defining $t_{n+1} = \lceil \frac{1}{b_n} \rceil$ is that this might be smaller than or equal to t_n. Note if $1/b_n > t_n$ for some n, then we have $t_{n+1} = \lceil \frac{1}{b_n} \rceil < 1 + \frac{1}{b_n}$ and

$$\frac{1}{b_{n+1}} = \frac{b_n + t_{n+1}}{b_n t_{n+1} - 1} > \frac{b_n + t_{n+1}}{b_n} > t_{n+1} t_n > t_{n+1}.$$

So we have $t_{n+k} = \lceil \frac{1}{b_{n+k}} \rceil$ for all $k \geq 1$ and we may use the above to arrive at contradiction. It remains to consider the case $1/b_n \leq t_n$ for all n. In this case, we have $t_{n+1} = t_n + 1 = t_1 + n$ for all n. However,

$$\sum_{i=1}^{\infty} \arctan\frac{1}{t_i} \gg \sum_{i=1}^{\infty} \frac{1}{t_1 + (i-1)} = \infty.$$