Week 2: Mock Putnam 2

1: Let k be a positive integer and let $S = \{n \in \mathbb{Z} \colon k^2 < n < (k+1)^2\}$. Prove that there does not exist distinct integers $a, b \in S$ such that ab is a perfect square.

2: Evaluate
$$\lim_{n\to\infty} \frac{1}{n} \int_0^n \frac{x \ln(1 + x/n)}{1 + x}\, dx.$$

3: For any positive integer n, let $d(n)$ denote the number of its positive divisors and let $\phi(n)$ denote the Euler-totient function of n (the number of integers $1, 2, \ldots, n$ coprime with n). Prove that
$$\sup_n \frac{d(\phi(n))}{d(n)} = \infty, \qquad \inf_n \frac{d(\phi(n))}{d(n)} = 0.$$

4: Let $f(x)$ be a polynomial with integer coefficients and let (a_n) be a strictly increasing sequence of positive integers such that $a_n \leq f(n)$ for all n. Prove that the set of primes dividing some a_n is infinite.

5: Prove that there exists a positive integer N such that for any integer $n > N$, there exists a finite set S of primes such that $n = \sum_{p \in S} \lfloor n/p \rfloor$.

6: Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + yf(x)) + f(xy) = f(x) + f(69y)$ for all $x, y \in \mathbb{R}$.

Week 2: Sketch of proofs

1: Let $a, b \in S$ such that ab is a perfect square and $a \neq b$. Let $d = \gcd(a, b)$; then we can write $a = dx$ and $b = dy$ for some positive integers x and y with $\gcd(x, y) = 1$. However, $ab = d^2xy$ is a square, so xy is a square. Since $\gcd(x, y) = 1$, both x and y must be squares. Let z and w be positive integers such that $x = z^2$ and $y = w^2$. Since $a \neq b$, we have $z \neq w$. The goal $a = b$ reduces to showing that $z = w$.

Notice that for any $n \in S$, we have $k < \sqrt{n} < k+1 \implies \lfloor \sqrt{n} \rfloor = k$. Since $a = dx = dz^2 \in S$ and $b = dy = dw^2 \in S$, we get $\lfloor z\sqrt{d} \rfloor = \lfloor w\sqrt{d} \rfloor = k$. The equality between floors yield $|z\sqrt{d} - w\sqrt{d}| < 1$. On the other hand,

$$|z\sqrt{d} - w\sqrt{d}| = |(z - w)\sqrt{d}| = |z - w|\sqrt{d}.$$

We have $\sqrt{d} \geq 1$, and since $z \neq w$, we also have $|z - w| \geq 1$. Thus $|z - w|\sqrt{d} \geq 1$; a contradiction.

2: **Answer.** $2 \ln 2 - 1$.

By the substitution $x = nt$, we get

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \frac{nt \ln(1 + t)}{1 + nt}\, ndt = \lim_{n\to\infty} \int_0^1 \frac{\ln(1 + t)}{1 + \frac{1}{nt}}\, dt.$$

Since $1 + \frac{1}{nt} \to 1$ as $n \to \infty$, one would expect

$$\lim_{n\to\infty} \int_0^1 \frac{\ln(1 + t)}{1 + \frac{1}{nt}}\, dt. = \int_0^1 \ln(1 + t)\, dt = \int_1^2 \ln t\, dt = [t \ln t - t]_1^2 = 2 \ln 2 - 1.$$

To prove the above equality, take the difference:

$$\Delta = \int_0^1 \ln(1 + t)\, dt - \lim_{n\to\infty} \int_0^1 \frac{\ln(1 + t)}{1 + \frac{1}{nt}}\, dt = \lim_{n\to\infty} \int_0^1 \frac{\ln(1 + t)}{1 + nt}\, dt.$$

For each n,

$$0 \leq \int_0^1 \frac{\ln(1 + t)}{1 + nt}\, dt \leq \int_0^1 \frac{t}{1 + nt}\, dt < \int_0^1 \frac{1}{n}\, dt = \frac{1}{n}.$$

By squeeze theorem, $\Delta = 0$ and we are done.

3: For the supremum, take n to be prime with $n \equiv 1 \pmod{2^k}$, and let $k \to \infty$. Note that by Dirichlet's theorem, such n exists for each k. Then

$$\frac{d(\phi(n))}{d(n)} = \frac{d(n-1)}{2} \geq \frac{k+1}{2},$$

and thus $\sup_n \dfrac{d(\phi(n))}{d(n)} = \infty$.

For the infimum, denote by $N\# = \prod_{p \leq N} p$ the primorial of N; here p ranges over all primes less than or equal to N. We take $n = (N\#)^M (2N)\#$, where M and N are arbitrary large positive integers. Then

$$d(n) = 2^{\pi(2N) - \pi(N)} (M + 2)^{\pi(N)}.$$

Since $N\#$ divides $(2N)\#$, we get

$$\phi(n) = (N\#)^M \phi((2N)\#) = (N\#)^M \prod_{p \le 2N} (p-1) = (N\#)^M 2^{\pi(2N)} \prod_{2<p\le 2N} \frac{p-1}{2}.$$

The terms in the product of the right hand side are pairwise distinct and less than N. Thus, $\phi(n)$ divides $(N\#)^M 2^{\pi(2N)} N!$, so

$$d(\phi(n)) \le d\left((N\#)^M 2^{\pi(2N)} N!\right).$$

For each prime p dividing $(N\#)^M 2^{\pi(2N)} N!$, it is easy to see that $p \le N$. Then $\nu_p(N\#) = 1$ and by Legendre's formula,

$$\nu_p(N!) = \frac{N - s_p(N)}{p-1} < N.$$

Thus we get the bound

$$\nu_p((N\#)^M 2^{\pi(2N)} N!) \le \begin{cases} M + \pi(2N) + N, & p = 2, \\ M + N, & p \ne 2. \end{cases}$$

Since $\pi(2N) \le N$ for N large enough, we get

$$d(\phi(n)) \le (M + \pi(2N) + N)(M+N)^{\pi(N)-1} \le 2(M+N)^{\pi(N)}.$$

As a result,

$$\frac{d(\phi(n))}{d(n)} \le \frac{1}{2^{\pi(2N)-\pi(N)-1}} \left(\frac{M+N}{M+2}\right)^{\pi(N)}.$$

For fixed N, letting $M \to \infty$ yields

$$\inf_n \frac{d(\phi(n))}{d(n)} \le \frac{1}{2^{\pi(2N)-\pi(N)-1}} \quad \forall N \in \mathbb{N}.$$

By Prime Number Theorem,

$$\pi(2N) - \pi(N) \sim \frac{N}{\ln N} \implies \lim_{N\to\infty} (\pi(2N) - \pi(N)) = \infty.$$

Thus $\inf_n \frac{d(\phi(n))}{d(n)} = 0$.

4: Let $k = \deg(f)$. Note that $k > 0$, since $(a_n)_{n\ge 1}$ is strictly increasing. There exists a positive integer N such that all coefficients of $(X+N)^k - f(X)$ are non-negative, and thus $f(n) \le (n+N)^k$ for all positive integers n.

Suppose for the sake of contradiction that the set S of primes dividing some a_n is finite. Let T be the set of positive integers whose all its prime divisors belong in S. Then $\{a_n : n \ge 1\} \subseteq T$. Since $a_n \le f(n) \le (n+N)^k$, we get the inequality

$$\sum_{n=1}^{\infty} \frac{1}{n+N} \le \sum_{n=1}^{\infty} \frac{1}{a_n^{1/k}} \le \sum_{x\in T} \frac{1}{x^{1/k}} = \prod_{p\in S} \sum_{i=1}^{\infty} \frac{1}{p^{i/k}} = \prod_{p\in S} \frac{1}{1-p^{-1/k}}.$$

The leftmost side is infinite, while the rightmost side is finite since S is finite. Contradiction.

5: We prove an alternate statement first: for any $0 < m \le n$, there exists a finite set S_m of primes less than or equal to n with the following properties:

- $|S_m| \le \log_2 m$;
- for each $k > 0$, there exists at most one prime $p \in S_m$ such that $\lfloor n/p \rfloor = k$;
- for each $m \le n$, the integer $\Delta_m := m - \sum_{p \in S_m} \lfloor n/p \rfloor$ satisfies $0 \le \Delta_m \le \log_2 m + 1$.

Proceed by induction on m. For the base case $m = 1$, just take $S_1 = \emptyset$.

Now we proceed for the induction step. Suppose that $m > 1$ and the above statement holds for all lesser m. By Bertrand's postulate, there exists a prime p_0 such that

$$\frac{n}{m+1} < p_0 \le \frac{2n}{m+1} \iff (m+1)/2 \le n/p_0 < m+1 \implies \lfloor (m+1)/2 \rfloor \le \lfloor n/p_0 \rfloor \le m.$$

Let $m_0 = m - \lfloor n/p_0 \rfloor$. The above bound gives $\lfloor n/p_0 \rfloor \ge m/2$. If $m_0 = 0$, then $S_m = \{p_0\}$ works. If $0 < m_0 < m/2$, then $m_0 < \lfloor n/p_0 \rfloor$, so we claim that $S_m = \{p_0\} \cup S_{m_0}$ works. Indeed, we have $\Delta_m = \Delta_{m_0}$, $\lfloor n/p_0 \rfloor > m_0 \ge \lfloor n/p \rfloor$ for all $p \in S_{m_0}$, and

$$|S_m| = 1 + |S_{m_0}| \le 1 + \log_2 m_0 \le 1 + \log_2(m/2) = \log_2 m.$$

Otherwise, we take $S_m = \{p_0\} \cup S_{m/2-1}$ instead. By the same argument as before, the first two properties hold on S_m. For the third property, note that

$$\Delta_m = m - \lfloor n/p_0 \rfloor - \sum_{p \in S_{m/2-1}} \lfloor n/p \rfloor = m/2 - \sum_{p \in S_{m/2-1}} \lfloor n/p \rfloor = \Delta_{m/2-1} + 1.$$

The bound $\Delta_{m/2-1} \le \log_2(m/2 - 1) + 1 \le \log_2 m$ yields $\Delta_m \le \log_2 m + 1$.

Now we go back to the main problem. For N large enough and $n \ge N$, the prime number theorem yields that there exists at least $\log_2 n + 2$ primes in the interval $(n/2, n)$. Then there exists at least $\log_2 n + 1 \ge \Delta_n$ such primes not in S_n. Let T denote the set of these Δ_n primes. Note that $\lfloor n/p \rfloor = 1$ for each $p \in T$. By choice of T, it is disjoint with S_n, so $S := S_n \cup T$ works.

6: Answer. Constant functions, $x \mapsto 69 - x$, and $x \mapsto \begin{cases} 0, & x \neq 0, \\ \alpha, & x = 0, \end{cases}$ for any constant $\alpha \in \mathbb{R}$.

For convenience, write $C = 69$. Clearly, constant functions work. On the other hand, if $f(x) = C - x$ for all $x \in \mathbb{R}$, then

$$f(x + yf(x)) + f(xy) = 2C - (x + y(C - x) + xy) = 2C - (x + Cy) = f(x) + f(Cy).$$

Thus, $x \mapsto C - x$ works as well.

Meanwhile, suppose that $f(x) = 0$ if $x \neq 0$ and $f(0) \neq 0$. If $x \neq 0$, then since $C \neq 0$, no matter whether y equals 0 or not,

$$f(x + yf(x)) + f(xy) = f(x) + f(xy) = f(x) + f(Cy).$$

If $x = 0$, then instead

$$f(x + yf(x)) + f(xy) = f(yf(0)) + f(0) = f(0) + f(f(0)y) = f(0) + f(Cy).$$

It remains to show that no other functions work. We may assume that f is non-constant and $f(x_0) \neq C - x_0 \iff x_0 + f(x_0) \neq C$ for some $x \in \mathbb{R}$. Our goal is to show that $f(x) = 0$ for all $x \in \mathbb{R}$ such that $x \neq 0$.

Plugging $x = C$ into the original equality gives $f(C + yf(C)) = f(C)$ for all $y \in \mathbb{R}$. If $f(C) \neq 0$, then plugging $y = \frac{z-C}{f(C)}$ into this equality gives $f(z) = f(C)$ for every $z \in \mathbb{R}$, and so f is constant; contradiction. Thus, $f(C) = 0$.

Plugging $y = 1$ into the original equation gives $f(x + f(x)) = f(C) = 0$ for all $x \in \mathbb{R}$. Recall by our assumption that $x_0 + f(x_0) \neq C$ for some $x_0 \in \mathbb{R}$. Thus, there exists $\alpha \neq C$ such that $f(\alpha) = 0$.

Plugging $x = \alpha$ into the original equality yields $f(\alpha y) = f(Cy)$ for all $y \in \mathbb{R}$. Replacing y with y/C yields $f(\beta y) = f(y)$ for all $y \in \mathbb{R}$, where $\beta = \alpha/C$. Note that $\beta \neq 1$, since $C \neq \alpha$.

Replacing x with βx in the original equality and using the above equality gives us

$$f(\beta x + yf(x)) + f(xy) = f(\beta x) + f(Cy) = f(x) + f(Cy) = f(x + yf(x)) + f(xy).$$

Thus $f(\beta x + yf(x)) = f(x + yf(x))$ for all $x, y \in \mathbb{R}$. Finally, we show that $f(x) = 0$ for all $x \neq 0$.

Suppose that there exists $x_0 \in \mathbb{R}$ such that $x_0 \neq 0$ and $f(x_0) \neq 0$. Then substituting $(x, y) = (x_0, (z - x_0)/f(x_0))$ into the above equation gives $f(\gamma + z) = f(z)$ for all $z \in \mathbb{R}$, where $\gamma = (\beta - 1)x_0$ is a non-zero real number. In short, f is periodic. But then for any $x, y \in \mathbb{R}$, we have

$$f(x + \gamma + yf(x + \gamma)) + f((x + \gamma)y) = f(x + \gamma) + f(Cy) = f(x) + f(Cy) = f(x + yf(x)) + f(xy).$$

Using periodicity reduces this equality to $f(xy + \gamma y) = f(xy)$. Plugging $(x, y) = (0, z/\gamma)$ yields $f(z) = f(0)$ for all $z \in \mathbb{R}$, so f is constant; a contradiction. We are done.

Note. The above solution works if we replace $\mathbb{R}$ with an arbitrary field and we replace $C = 69$ with an arbitrary non-zero element of the field. If $C = 0$, then the only difference is that the third family of solutions do not work anymore.