Week 3: Mock Putnam 3

1: What are the possible values for $\sum_{n=1}^{\infty} x_n^{69}$ where $\{x_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} x_n = 2024$?

2: Let R be a ring with unity such that for every element $a \in R$, either $a^2 = 1$ or $a^n = 0$ for some positive integer n. Prove that R is a commutative ring.

3: Let (a_n) be a sequence of real numbers such that

$$\lim_{n\to\infty} a_{n+1} - a_n = 0, \qquad \lim_{n\to\infty} a_{2n} - 2a_n = 0.$$

Prove that $\lim_{n\to\infty} a_n = 0$.

4: Let G be a graph with vertices labeled $1, 2, \ldots, n$ and degrees (the number of vertices with an edge to it) $d_1, \ldots, d_n$. For any permutation σ on $\{1, \ldots, n\}$, we say i is a local minimum of σ if $\sigma(i) < \sigma(j)$ for any vertex j with an edge to i. Find the expected number of local minima over all permutations σ. (Note: this gives a lower bound for the size of a maximal subset of vertices with no edges between any two, also known as an *independent set.*)

5: Let $n \geq 1$ be an integer. Let $p(x)$ be a real polynomial of degree n and let $a \geq 3$ be a real number. Prove that

$$\max_{k=0,1,\ldots,n+1} |a^k - p(k)| \geq 1.$$

6: Let A, B be 69×69 singular matrices with coefficients in $\mathbb{C}$. Suppose $(AB)^{69} = 0$ and $(AB)^{68} \neq 0$. Prove that $(BA)^{68} = 0$.

Week 3: Sketch of proofs

1: Answer. The set of such values is the open interval $(0, 2024^{69})$.

To minimize the sum value, we want all the x_ns to be small simultaneously. To maximize the sum value, we want x_1 to be very big, close to 2024. We now write down this idea formally.

First, it is clear that the given expression must be positive. Second, for any $a, b \in \mathbb{R}^+$, we have $(a+b)^{69} > a^{69} + b^{69}$, so

$$\sum_{n=1}^{\infty} x_n^{69} < \left(\sum_{n=1}^{\infty} x_n\right)^{69} = 2024^{69}.$$

It remains to show that all positive values less than 2024^{69} are attainable.

For each positive integer N, define the function $f_N : [0, 2024] \to \mathbb{R}$ by

$$f_N(x) = \sum_{n=1}^{N} \left(\frac{2024 - x}{N}\right)^{69} + \sum_{n=N+1}^{\infty} \left(\frac{x}{2^{n-N}}\right)^{69} = \frac{(2024 - x)^{69}}{N^{68}} + \frac{x^{69}}{2^{69} - 1}.$$

For $x \in (0, 2024)$, the numbers $(2024 - x)/N$ and $x/2^{n-N}$ are positive for all $n \geq N + 1$, and

$$\sum_{n=1}^{N} \frac{2024 - x}{N} + \sum_{n=N+1}^{\infty} \frac{x}{2^{n-N}} = N \cdot \frac{2024 - x}{N} + \sum_{n=1}^{\infty} \frac{x}{2^n} = (2024 - x) + x = 2024.$$

Thus, for each $x \in (0, 2024)$, $f_N(x)$ is a possible value taken by the given expression. We now find values attained by each f_N on $(0, 2024)$.

Notice that $f_N(0) = \dfrac{2024^{69}}{N^{68}}$ and $f_N(2024) = X$, where $X = \frac{2024^{69}}{2^{69}-1}$. For $N = 1$, we have $f_N(0) > f_N(2024)$, so numbers in $(X, 2024^{69})$ are attainable. For $N > 1$, we have $f_N(0) < f_N(2024)$, so numbers in $(2024^{69}/N^{68}, X)$ are attainable. Letting $N \to \infty$, all positive values less than X are attainable. It remains to see if X is attainable. Indeed, we can take each $x_n = 2024/2^n$ in this case. By geometric series formula,

$$\sum_{n=1}^{\infty} x_n = 2024, \quad \sum_{n=1}^{\infty} x_n^{69} = \sum_{n=1}^{\infty} \frac{2024^{69}}{2^{69n}} = \frac{2024^{69}}{2^{69} - 1} = X.$$

2: Assume that R is non-trivial; $1 \neq 0$ in R. We first claim that $a^2 = 1$ if and only if a is a unit. Clearly, if $a^2 = 1$, then a is a unit (with itself as inverse). Conversely, suppose for the sake of contradiction that a is a unit, say with inverse b, and $a^n = 0$ for some $n > 0$. By induction on k, it is easy to see that $b^k a^k = 1$. But then $1 = b^n a^n = b^n 0 = 0$; contradiction. This proves the claim.

Now, for any unit a and b, ab is also a unit. By the previous paragraph, $a^2 = b^2 = (ab)^2 = 1$. But then $ab = a(ab)^2 b = a^2 bab^2 = ba$. Thus, every two unit in R commutes. Note that if a commute with b, then both a and $1 - a$ commute with both b and $1 - b$. It remains to show the following: if a is not a unit, then $1 - a$ is a unit.

Suppose for the sake of contradiction that both a and $1-a$ are not units for some $a \in R$. That is, $a^n = 0$ and $(1-a)^m = 0$ for some $m, n > 0$. The second equality yields

$$0 = (1-a)^m (a^{n-1} + a^{n-2} + \ldots + 1)^m = ((1-a)(a^{n-1} + \ldots + 1))^m = (1-a^n)^m = 1.$$

Contradiction; thus a and $1-a$ are units. We are done.

3: We first show that

$$\lim_{n \to \infty} \frac{a_n}{n} = 0.$$

Fix any $\varepsilon > 0$. The first limit condition implies that there exists $N > 0$ such that $|a_{n+1} - a_n| < \varepsilon/2$ for all $n \geq N$. By triangle inequality and induction, we get

$$|a_{n+N}| \leq |a_N| + \frac{\varepsilon}{2} n \quad \forall n \geq 0.$$

For $n > 2\varepsilon^{-1}|a_N|$, we get $|a_{n+N}| < \varepsilon n$. This shows the above limit equality.

Now we show that $a_n \to 0$ as $n \to \infty$. Fix an arbitrary $\varepsilon > 0$. Let N be a positive integer such that $|a_{2n} - 2a_n| < \varepsilon$ for all $n \geq N$. We claim that $|a_n| \leq \varepsilon$ for any $n \geq N$. Indeed, by induction on k, for any $n \geq N$ and $k \geq 0$,

$$|a_{2^k n}| > 2^k |a_n| - (2^k - 1)\varepsilon > 2^k(|a_n| - \varepsilon) \implies \frac{|a_{2^k n}|}{2^k n} > \frac{|a_n| - \varepsilon}{n}.$$

Taking limit as $k \to \infty$ yields the claim.

4: Answer. $\displaystyle\sum_{i=1}^{n} \frac{1}{d_i + 1}$.

For each $i \leq n$, let $X_i := X_i(\sigma)$ denote the random variable that is equal to 1 if i is a local minimum of σ and 0 otherwise. Then by linearity of expectation, the expected number of local minima is equal to

$$E[X_1 + X_2 + \ldots + X_n] = \sum_{i=1}^{n} E[X_i].$$

Now fix some $i \leq n$, and let $v_1, v_2, \ldots, v_{d_i}$ be the neighbours of i in G. Consider a fixed subset $S \subseteq \{1, 2, \ldots, n\}$ of size $d_i + 1$, and restrict our attention to permutations σ such that

$$\{\sigma(i)\} \cup \{\sigma(v_k) : k \leq d_i\} = S.$$

Then i is a local minimum iff $\sigma(i)$ is the minimal element of S. As σ ranges over all permutations satisfying the above equality, the probability that this happens is exactly $1/|S| = 1/(d_i+1)$. Letting S ranges over all subsets of size $d_i + 1$, we get that $E[X_i] = 1/(d_i + 1)$. We are done.

5: Suppose that the inequality is false. That is, assume for the sake of contradiction that $|a^k - p(k)| < 1$ for all $k = 0, 1, \ldots, n+1$. By the general theory of forward differencing, given a real polynomial p of degree n, the polynomial

$$q(x) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} p(x+k)$$

has degree exactly $n-m$ if $m \le n$, and is zero if $m > n$. To prove this statement, denote by $\Delta : \mathbb{R}[x] \to \mathbb{R}[x]$ the linear operator given by $p(x) \mapsto p(x) - p(x+1)$. Then the polynomial $q(x)$ defined above is exactly $\Delta^m(p)$. Now one can check that $\Delta(p)$ has degree exactly $\deg(p) - 1$ if p is non-constant and $\Delta(p) = 0$ if p is constant. Induction on m then proves the statement.

Applying forward differencing, we get

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} p(k) = 0,$$

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} (a^k - p(k)) = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} a^k = (1-a)^k,$$

and taking absolute value, we get

$$2^k = \sum_{k=0}^{n+1} \binom{n+1}{k} > \sum_{k=0}^{n+1} \binom{n+1}{k} |a^k - p(k)| \ge |(1-a)^k| = |1-a|^k \ge 2^k.$$

Contradiction.

6: Let $n = 69$, and let r be the rank of A. Note that $r < n$, since A is singular.

By Gaussian elimination, there exists $P \in M_n(\mathbb{C})$ invertible such that $P^{-1}A$ is in reduced row echelon form. By Gaussian elimination again, there exists $Q \in M_n(\mathbb{C})$ invertible such that $P^{-1}AQ^{-1}$ is diagonal with first r entries 1 and all other entries zero. Thus, we can write

$$A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q, \qquad B = Q^{-1} \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} P^{-1},$$

where $B_1 \in M_r(\mathbb{C})$. Then

$$AB = P \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix} P^{-1}, \qquad BA = Q^{-1} \begin{pmatrix} B_1 & 0 \\ B_3 & 0 \end{pmatrix} Q.$$

By induction on k, we get

$$(AB)^k = P \begin{pmatrix} B_1^k & B_1^{k-1}B_2 \\ 0 & 0 \end{pmatrix} P^{-1}, \qquad (BA)^k = Q^{-1} \begin{pmatrix} B_1^k & 0 \\ B_3 B_1^{k-1} & 0 \end{pmatrix} Q.$$

Since $(AB)^n = 0$, we have $B_1^n = 0$, so B_1 is nilpotent. Since $(AB)^{n-1} \ne 0$, we have $B_1^{n-2}B_2 \ne 0$. In particular, $B_1^{n-2} \ne 0$, so the minimal polynomial of B_1 is X^k with $k \in \{n-1, n\}$. Since B_1 has dimension $r < n$, this necessarily implies $r = k = n-1$. In particular, we have a Jordan decomposition for B_1 of form SJS^{-1} for some invertible matrix S, where $J \in M_{n-1}(\mathbb{C})$ is defined by

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

In particular, J^{n-2} is the $(n-1)\times(n-1)$-matrix with $(1,n-1)$-entry 1 and all other entries zero.

Recall that $B_1^{n-2}B_2 \neq 0$. The condition $(BA)^{n-1} = 0$ is equivalent to $B_3B_1^{n-2} = 0$ since we know $B_1^{n-1} = 0$. We now change basis again for B; we can write

$$B = Q^{-1}\begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} J & \mathbf{x} \\ \mathbf{y}^T & c \end{pmatrix}\begin{pmatrix} S^{-1} & 0 \\ 0 & 1 \end{pmatrix}P^{-1}$$

for some $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{n-1}$ and $c \in \mathbb{C}$. Here $B_2 = S\mathbf{x}$ and $B_3 = \mathbf{y}^T S^{-1}$. Thus we have

$$B_1^{n-2}B_2 \neq 0 \iff J^{n-2}\mathbf{x} \neq 0 \iff x_{n-1} \neq 0.$$

The condition that B is singular is equivalent to $\begin{pmatrix} J & \mathbf{x} \\ \mathbf{y}^T & c \end{pmatrix}$ being singular. By the form J takes, this matrix has determinant $-x_{n-1}y_1$. Since $x_{n-1} \neq 0$, singularity of this matrix implies $y_1 = 0$.

Finally, notice that the goal reduces to

$$B_3B_1^{n-2} = 0 \iff \mathbf{y}^T J^{n-2} = 0 \iff y_1 = 0.$$

This shows that $B_3B_1^{n-2} = 0$, and thus $(BA)^{n-1} = 0$.