Week 4: Mock Putnam 4

1: Prove that the equation

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} = \frac{1}{69}$$

has finitely many positive integer solutions.

2: Let S be the set of all $n \times n$ matrices with coefficients in $\mathbb{Z}/69\mathbb{Z}$. Prove that

$$\sum_{A \in S} \mathrm{Tr}(A^2) = \sum_{A \in S} (\mathrm{Tr}(A))^2.$$

3: (a) Let $v_1, \ldots, v_k \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\langle v_i, v_j \rangle \leq 0$ for all $i \neq j$, where $\langle \, , \rangle$ denotes the standard inner product. Prove that $k \leq 2n$.

(b) An $m \times n$ matrix is filled with 0 and 1 such that any two rows differ in at least $n/2$ positions. Prove that $m \leq 2n$.

4: Let A and B be two finite sets of positive real numbers such that

$$\left\{ \sum_{x \in A} x^n : n \in \mathbb{N} \right\} \subseteq \left\{ \sum_{x \in B} x^n : n \in \mathbb{N} \right\}.$$

Prove that there exists $k \in \mathbb{Z}$ such that $A = \{x^k : x \in B\}$.

5: Let n, k be positive integers such that $n \geq 2k$. Let S be a collection of pairwise non-disjoint k-element subsets of $\{1, 2, \ldots, n\}$. Prove that $|S| \leq \binom{n-1}{k-1}$.

6: Let $a \leq b$ be real numbers. Let $f : [a, b] \to [a, b]$ satisfy $|f(x) - f(y)| \leq |x - y|$ for any $x, y \in [a, b]$. Let $x_1 \in [a, b]$ and $x_{n+1} = \dfrac{x_n + f(x_n)}{2}$. Prove that $\lim_{n \to \infty} x_n$ exists.

Week 4: Sketch of proofs

1: Let $\alpha = 1/69 > 0$. If a triple (x, y, z) work, any permutations of (x, y, z) also work. Thus the total number of triples (x, y, z) satisfying the equality is at most 6 times the number of triples satisfying the equality with an added condition: $x \geq y \geq z$. It suffices that the equation has finitely many positive integer solutions with $x \geq y \geq z$.

Notice that the function $f(x) = \frac{x}{1+x^2} = \frac{1}{x+1/x}$ is strictly decreasing on the positive integers. Thus $f(x) \leq f(y) \leq f(z)$, and $f(x)+f(y)+f(z) \leq \alpha$ yields $f(z) \geq \alpha/3$. Since f is decreasing and $f(x) \to 0$ as $x \to \infty$, there are only finitely many choices of z satisfying $f(z) \geq \alpha/3$.

Next, for fixed $z = z_0$, we have $f(x) + f(y) = \alpha - f(z_0) \geq 2\alpha/3 > 0$. Since $f(x) \leq f(y)$, we get $f(y) \geq (\alpha - f(z_0))/2 > 0$. By the same argument as before, there are only finitely choices of y that works.

Finally, since f is strictly decreasing, it is injective. Thus for any fixed z and y, there exists at most one positive integer x such that $f(x) = \alpha - f(y) - f(z)$. Since we also only have finitely many choices for y for a given z and also finitely many choices for z, we have finitely many triplets (x, y, z) such that $f(x) + f(y) + f(z) = \alpha$ and $x \geq y \geq z$.

2: Write $R = \mathbb{Z}/69\mathbb{Z}$. The case $n = 1$ is straightforward, so we assume that $n \geq 2$. Then we claim that both sides are actually zero.

For each $A \in S$ and $i, j \leq n$, denote by $A_{ij} \in R$ the (i, j)-entry of A. Notice that

$$(A^2)_{ii} = \sum_{j=1}^{n} A_{ij}A_{ji} \ \forall i \leq n \implies \mathrm{Tr}(A^2) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}A_{ji}.$$

The left hand side becomes

$$\sum_{A\in S} \mathrm{Tr}(A^2) = \sum_{A\in S}\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}A_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{A\in S} A_{ij}A_{ji}.$$

For fixed i and j, there are $|R|^{n^2-2}$ choices for entries of A other than A_{ij} and A_{ji} if $i \neq j$, and $|R|^{n^2-1}$ choices if $i = j$. There are $n(n-1)$ pairs (i, j) with $i \neq j$ and n pairs with $i = j$, so

$$\sum_{A\in S} \mathrm{Tr}(A^2) = n(n-1)|R|^{n^2-2} \sum_{a,b\in R} ab + n|R|^{n^2-1} \sum_{a\in R} a^2.$$

However, $|R| = 0$ in R and $n^2 - 1, n^2 - 2 > 0$, so the right hand side is zero.

On the other hand, since $\mathrm{Tr}(A)$ only depends on the diagonal entries of A and there are $n(n-1)$ non-diagonal entries,

$$\sum_{A\in S} (\mathrm{Tr}(A))^2 = |R|^{n(n-1)} \sum_{a_1,\ldots,a_n\in R} (a_1 + a_2 + \ldots + a_n)^2.$$

Since $n(n-1) > 0$, this equals zero, as desired.

3: For part (a), we prove the contrapositive. It suffices to consider the case $k = 2n + 1$. That is, we prove that for any $v_1, \ldots, v_{2n+1} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, there exists $i \neq j$ such that $\langle v_i, v_j \rangle > 0$. Proceed by induction on n.

For the base case $n = 1$, we have $v_1, v_2, v_3 \in \mathbb{R}$ non-zero. Then two of v_1, v_2, v_3 has the same sign, and thus their product is positive.

Now we proceed for the induction step. Suppose that the induction hypothesis holds for some n. Consider arbitrary vectors $v_1, v_2, \ldots, v_{2n+3} \in \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$. By rotation and scaling, we can assume WLOG that $v_{2n+3} = e_{n+1} = (0, 0, \ldots, 0, -1)$. For each $i \leq 2n + 2$, write $v_i = (w_i, c_i)$ for some $w_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$, with either $w_i \neq \mathbf{0}$ or $c_i \neq 0$. Then $\langle v_i, v_{2n+3} \rangle = -c_i$, so we are done if $c_i < 0$ for some $i \leq 2n + 2$. We now assume that $c_i \geq 0$ for all $i \geq 2n + 2$.

If $w_i = w_j = \mathbf{0}$ for some $i \neq j$, then $c_i, c_j > 0$. But then $\langle v_i, v_j \rangle = \langle (\mathbf{0}, c_i), (\mathbf{0}, c_j) \rangle = c_i c_j > 0$. The remaining case is where $w_i = \mathbf{0}$ for at most one value of $i \leq 2n + 2$. By rearranging, we may assume that $w_i \neq \mathbf{0}$ for all $i \leq 2n + 1$. By induction hypothesis, there exists $i < j \leq 2n + 1$ such that $\langle w_i, w_j \rangle > 0$. Finally, this means

$$\langle v_i, v_j \rangle = \langle w_i, w_j \rangle + c_i c_j > 0 + 0 = 0.$$

Induction step is complete. Part (a) is proved.

For part (b), let $v_1, \ldots, v_m$ be the rows of the matrix, except that all occurrences of 0 are replaced with occurrences of -1. Since all entries of each v_i are ± 1, we get

$$\langle v_i, v_j \rangle = \sum_{k=1}^{n} (v_i)_k (v_j)_k = 1 \cdot |\{k : (v_i)_k = (v_j)_k\}| + (-1) \cdot |\{k : (v_i)_k = -(v_j)_k\}| \quad \forall i, j \leq n.$$

By the problem's assumption, if $i \neq j$, then $(v_i)_k = (v_j)_k$ for at most $n/2$ indices k, and $(v_i)_k = -(v_j)_k$ for at least $n/2$ indices k. Thus $\langle v_i, v_j \rangle \leq 0$ for all $i \neq j$. By part (a), we get $m \leq 2n$.

4: For convenience, for any finite subset $C \subseteq \mathbb{R}^+$ and $n \in \mathbb{N}$, denote $\sigma_n(C) = \sum_{x \in C} x^n$. Note that $\sigma_n(C) = 0$ if and only if $C = \emptyset$. We start by making some general observations.

- Let $A \subseteq \mathbb{R}^+$ be non-empty finite and let a_0 be the largest element of A. Then for any $n \in \mathbb{N}$,

  $$\frac{\sigma_n(A)}{a_0^n} = \sum_{a \in A} \left(\frac{a}{a_0} \right)^n \xrightarrow{n \to \infty} 1,$$

  since $a/a_0 \leq 1$, with equality iff $a = a_0$. In particular, for any $\varepsilon > 0$, there exists a positive integer N such that $1 \leq \sigma_n(A)/a_0^n < 1 + \varepsilon < e^\varepsilon$ for any $n \geq N$. Taking logarithm yields

  $$0 \leq \ln \sigma_n(A) - n \ln a_0 < \varepsilon \quad \forall n \geq N.$$

- Let $A \subseteq \mathbb{R}^+$ be non-empty finite and let a_0 be the largest element of A. If $a_0 \leq 1$ and $A \neq \{1\}$, then the sequence $(\sigma_n(A))_{n \geq 1}$ is strictly decreasing. The sequence converges to 1 if $a_0 = 1$ and 0 if $a_0 < 1$. On the other hand, by the previous observation, this sequence is unbounded and in fact diverges to $+\infty$ if $a_0 > 1$. In particular, for each $M > 0$, there exists only finitely many n such that $\sigma_n(A) \leq M$.

Now we go back to the main problem. If $B = \emptyset$, then $\sigma_n(B) = 0$ for all $n \geq 1$. This would imply $\sigma_n(A) = 0$ for all $n \geq 1$, so $A = \emptyset$ and we are done. Thus, we now assume that $B \neq \emptyset$, which also means $A \neq \emptyset$. If $A = \{1\}$, we are done since $A = \{x^0 : x \in B\}$. We now assume $A \neq \{1\}$ and prove a stronger statement: there exists $m \in \mathbb{N}$ such that $A = \{x^m : x \in B\}$. Let a_0 be the largest element of A and b_0 be the largest element of B.

If $a_0 > 1$, then $(\sigma_n(A))_{n\geq 1}$ is unbounded. Thus, $(\sigma_n(B))_{n\geq 1}$ is also unbounded, yielding $b_0 > 1$. On the other hand, if $a_0 \leq 1$, then since $A \neq \{1\}$, the sequence $(\sigma_n(A))_{n\geq 1}$ is strictly decreasing. Recall that for each n, there exists k such that $\sigma_n(A) = \sigma_k(B)$. In particular, infinitely many terms of the sequence $(\sigma_n(B))_{n\geq 1}$ are less than or equal to $\sigma_1(A)$. This forces $b_0 \leq 1$, and $B \neq \{1\}$ since $(\sigma_n(B))_{n\geq 1}$ contains infinitely many distinct real numbers, namely, the terms of $(\sigma_n(A))_{n\geq 1}$. If $a_0 = 1$, then $(\sigma_n(B))_{n\geq 1}$ contains infinitely many terms greater than 1, so $b_0 = 1$. If $a_0 < 1$, then $(\sigma_n(B))_{n\geq 1}$ contains a term less than 1, forcing $b_0 < 1$. In summary, we have either $a_0 = b_0 = 1$ or $a_0, b_0 > 1$ or $a_0, b_0 < 1$. Note that the latter two implies that $\ln a_0 / \ln b_0$ is positive.

If $a_0 = b_0 = 1$, then for each n, there exists k such that

$$\sigma_n(A \setminus \{1\}) = \sigma_n(A) - 1 = \sigma_k(B) - 1 = \sigma_k(B \setminus \{1\}).$$

If $A \setminus \{1\} = \{x^m : x \in B \setminus \{1\}\}$ for some $m \in \mathbb{Z}$, then $A = \{x^m : x \in B\}$. Thus, it remains to solve the case where $a_0, b_0 \neq 1$.

For convenience, write $m = \ln a_0 / \ln b_0 > 0$. Let $\varepsilon = |\ln b_0|/4$. By the first observation, there exists N such that for any $n \geq N$,

$$0 \leq \ln \sigma_n(A) - n \ln a_0 < \varepsilon \quad \text{and} \quad 0 \leq \ln \sigma_n(B) - n \ln b_0 < \varepsilon.$$

Recall that for each n, there exists $k := k(n)$ such that $\sigma_n(A) = \sigma_{k(n)}(B)$. If $a_0, b_0 < 1$, then for n large enough, $\sigma_n(A) \leq \sigma_N(B)$, which forces $k \geq N$. If $a_0, b_0 > 1$, then for n large enough, $\sigma_n(A) > \sup_{i \leq N} \sigma_i(B)$, which also forces $k \geq N$. Thus, there exists $N_0 > N$ such that $k(n) \geq N$ for all $n \geq N_0$. In particular, we get $0 \leq \ln \sigma_n(A) - n \ln a_0 < \varepsilon$ and $0 \leq \ln \sigma_n(A) - k(n) \ln b_0 < \varepsilon$, so

$$|n \ln a_0 - k(n) \ln b_0| < \varepsilon = |\ln b_0|/4 \implies |nm - k(n)| < 1/4.$$

In summary, there exists a positive integer N_0 such that nm is at distance at most $1/4$ from an integer for all $n \geq N_0$. If $m = \ln a_0 / \ln b_0$ is irrational, then Kronecker's approximation theorem yields a contradiction. Thus, m is rational.

We now prove that m is an integer. Write $m = p/q$, where p and q are coprime positive integers. If q is even, then there exists $n \geq N_0$ such that $np \equiv q/2 \pmod{q}$. Then the fractional part of np/q is $1/2$, which means $|np/q - k| \geq 1/2$ for any integer k; contradiction. If $q > 1$ is odd, then $n \geq N_0$ such that $np \equiv (q-1)/2$. Then the fractional part of np/q is $(q-1)/(2q)$, which means $|np/q - k| \geq (q-1)/(2q) \geq 1/3$ for any integer k; contradiction. Thus $q = 1$, and so m is indeed an integer.

Recall that for any $n \geq N_0$, we have $|nm - k(n)| < 1/4$. Since m and $k(n)$ are integers, we get $k(n) = nm$ for each $n \geq N_0$. That is, for any $n \geq N_0$,

$$\sigma_n(A) = \sigma_{nm}(B) = \sigma_n(\{x^m : x \in B\}).$$

It remains to show the following claim: for any $A_1, A_2 \subseteq \mathbb{R}^+$ finite, if $\sigma_n(A_1) = \sigma_n(A_2)$ for all n large enough, say $n \geq N$, then $A_1 = A_2$. By induction on say $\max\{|A_1|, |A_2|\}$, the claim further reduces to the following statement: given A_1 and A_2 as above, if both are non-empty, then the largest element of the two sets are equal. Note that A_1 and A_2 are either both empty or both non-empty.

Let a_1 and a_2 be the largest element of A_1 and A_2, respectively. WLOG assume that $a_1 \leq a_2$, and suppose for the sake of contradiction that $a_1 < a_2$. Then for n large enough, we have $a_2^n > 2a_1^n$. But also for n large enough, we have $\sigma_n(A_1) < 2a_1^n$. Then for any n large enough, we get

$$\sigma_n(A_1) < 2a_1^n < a_2^n \leq \sigma_n(A_2),$$

contradiction. The claim is proved.

5: We do some sort of double-counting. Let S_n denote the symmetric group on $\{1, 2, \ldots, n\}$. For each $\sigma \in S_n$ and $0 \leq i < n$, denote $T_{\sigma,i} = \{\sigma(i + j) : 0 \leq j < k\}$, where the indices are taken mod n (so $i + j$ actually means $i + j - n$ if $i + j > n$). Denote $A(\sigma) = \{i : T_{\sigma,i} \in S\}$.

First consider an arbitrary $T \in S$ and an index $i < n$. Since $|T| = k$, there exists $k!$ ways to arrange the elements of T in an ordered k-tuple. For each ordered k-tuple $(a_0, a_1, \ldots, a_{k-1})$, there exists $(n - k)!$ permutations σ such that $a_j = \sigma(i + j)$ for each $j = 0, 1, \ldots, k - 1$. Thus for each i, there exists $k!(n - k)!$ permutations $\sigma \in S_n$ such that $T_{\sigma,i} = T$. By double-counting, we get

$$\sum_{\sigma \in S_n} |A(\sigma)| = \#\{(\sigma, i) : T_{\sigma,i} \in S\} = k!(n - k)! \cdot n \cdot |S|.$$

On the other hand, we show that $|A(\sigma)| \leq k$ for any $\sigma \in S_n$. Fix σ and $i_0 \in A(\sigma)$, if any. For any $i \in A(\sigma)$, note that $T_{\sigma,i} \cap T_{\sigma,i_0} \neq \emptyset$. Thus there exists $0 \leq j_1, j_2 < k$ such that

$$\sigma(i + j_1) = \sigma(i_0 + j_2) \iff i + j_1 = i_0 + j_2 \iff i - i_0 = j_1 - j_2 \in \{0, \pm 1, \pm 2, \ldots, \pm(k - 1)\}.$$

That is,

$$A(\sigma) \subseteq \{i_0, i_0 \pm 1, \ldots, i_0 \pm (k - 1)\}.$$

Now suppose for the sake of contradiction that $|A(\sigma)| \geq k + 1$. Then $A(\sigma)$ contains at least k elements aside from i_0. By the above argument and pigeonhole principle, there exists $j \in \{1, 2, \ldots, k - 1\}$ such that $i_0 + j, i_0 - (k - j) \in A(\sigma)$. But $(i_0 + j) - (i_0 - (k - j)) = k$, not congruent to any of $0, \pm 1, \ldots, \pm(k - 1)$ mod n since $n \geq 2k$; contradiction. This proves that $|A(\sigma)| \leq k$ for any $\sigma \in S_n$.

As a result, we get

$$k!(n - k)! \cdot n \cdot |S| = \sum_{\sigma \in S_n} |A(\sigma)| \leq |S_n| k = n! \cdot k \implies |S| \leq \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}.$$

6: Define the function $g(x) = \dfrac{x + f(x)}{2}$. Note that for any $x \in [a, b]$, since $f(x) \in [a, b]$, we also have $g(x) \in [a, b]$. We claim that g is non-decreasing. Indeed, fix $x, y \in [a, b]$ such that $x \leq y$. Then

$$g(y) - g(x) = \frac{y - x + f(y) - f(x)}{2} \geq \frac{|y - x| - |f(y) - f(x)|}{2} \geq 0.$$

This proves the claim.

Now we go back to the main problem. Recall that $x_{n+1} = g(x_n)$ for each $n \geq 1$. By induction, we get $x_{n+1} = g^n(x_1)$ for each $n \geq 0$. If $x_1 \leq x_2 = g(x_1)$, then we get

$$x_1 \leq g(x_1) \leq g^2(x_1) \leq \dots.$$

That is, by induction on n, we get $x_{n+1} \geq x_n$ for all $n \geq 1$. The sequence $(x_n)_{n \geq 1}$ is non-decreasing, but bounded above by b, so it converges by monotone convergence theorem.

On the other hand, if $x_1 \geq x_2 = g(x_1)$, then

$$x_1 \geq g(x_1) \geq g^2(x_1) \geq \dots.$$

That is, by induction on n, we get $x_{n+1} \leq x_n$ for all $n \geq 1$. The sequence $(x_n)_{n \geq 1}$ is non-increasing, but bounded below by a, so it converges by monotone convergence theorem.