Week 5: Mock Putnam 5

1: Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) \leq x$ and $f(x+y) \leq f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

2: Let $m \in \mathbb{N}$. We say a pair (i, j) with $1 \leq i < j \leq 4m+2$ is m-*admissible* if the set $\{1, 2, \ldots, 4m+2\} \backslash \{i, j\}$ can be partitioned into m 4-term arithmetic progressions. For example, $(2, 9)$ is 2-admissible using the partition $\{1, 3, 5, 7\}$ and $\{4, 6, 8, 10\}$. Prove that the number of m-admissible pairs is at least $m^2 + m + 1$. (Extra: Is it exactly $m^2 + m + 1$?)

3: Let $f : (0, \infty) \to (0, \infty)$ be a function such that $\dfrac{f(x) - 1}{x}$ is bounded below. Let $a_1 = 69$ and $a_{n+1} = a_n f(a_n)$ for $n \geq 1$. Prove that $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

4: For which primes p does there exist a group with an element a of order 11 and an element b of order p such that $ba = ab^2$?

5: Let m, n be relative prime integers. Let $\alpha > 1$ be a real number such that $\alpha^m + \alpha^{-m}$ and $\alpha^n + \alpha^{-n}$ are integers. Prove that $\alpha + \alpha^{-1}$ is also an integer.

6: Let n be a positive integer. A collection of $(n+1)^2$ points are placed inside a square of side length n. Prove that there exists three points such that the triangle determined by them has area at most $1/2$. (If the three points are collinear, the triangle has area 0.)

Week 5: Sketch of proofs

1: First notice that the two inequalities yield $f(0) \leq 0$ and $f(0) \leq f(0) + f(0)$. The latter yields $0 \leq f(0)$, and so we get $f(0) = 0$.

Now plug $y = -x$ into the second inequality, and get

$$0 = f(0) \leq f(x) + f(-x) \leq x + (-x) \leq 0.$$

This means equality must occur in all the inequalities, which forces $f(x) = x$ for all $x \in \mathbb{R}$. Finally, it is clear that $f(x) = x$ works. Thus, the answer is $\boxed{f(x) = x.}$

2: Working with the cases $m = 0, 1, 2, 3$ help finding m-admissible pairs.

We exhibit some m-admissible pairs and count their number. First, consider the pairs $(4a + 1, 4b + 2)$ for each $0 \leq a < b \leq m$. From the set $\{1, 2, \ldots, 4m + 2\} \setminus \{4a + 1, 4b + 2\}$, we take out the sets $\{4i + 1, 4i + 2, 4i + 3, 4i + 4\}$ for each $0 \leq i < a$ and $\{4k + 3, 4k + 4, 4k + 5, 4k + 6\}$ for each $b \leq k < m$. The remaining set is are m-admissible since

$$\{4a + 2, 4a + 3, \ldots, 4b + 1\} = \bigcup_{j=a}^{b-1} \{4j + 2, 4j + 3, 4j + 4, 4j + 5\}.$$

The $b - a$ sets are pairwise disjoint, say by order, the elements of each of them form a 4-term arithmetic progression of difference 1. The number of pairs counted here is

$$\binom{m+2}{2} = \frac{(m + 1)(m + 2)}{2}.$$

Next, as the example may suggest, we may also consider partitions into 4-term arithmetic progression of difference greater than 1. This time, consider the pairs $(4a + 2, 4(a + c) + 1)$ for some $c \geq 2$ and $0 \leq a \leq m - c$. Take out the sets $\{4i + 1, 1i + 2, 4i + 3, 4i + 4\}$ for each $0 \leq i < a$, and also take out the sets $\{4k + 3, 4k + 4, 4k + 5, 4k + 6\}$ for each $a + c \leq k < m$. The remaining set is

$$\{4a{+}1, 4a{+}3, 4a{+}4, \ldots, 4(a{+}c){+}4, 4(a{+}c){+}6\} = \bigcup_{i \in \{1,3,4,\ldots,c,c+2\}} \{4a{+}i, 4a{+}c{+}i, 4a{+}2c{+}i, 4a{+}3c{+}i\}.$$

The c sets are pairwise disjoint by mod c reason, and the elements of each of them form a 4-term arithmetic progression of difference c. Since the case $c = 1$ does not count, the number of pairs counted here is

$$\binom{m}{2} = \frac{m(m - 1)}{2}.$$

In total, we already get $\dfrac{(m + 1)(m + 2)}{2} + \dfrac{m(m - 1)}{2} = m^2 + m + 1$ pairs.

Solution for the extra problem. For m large enough, these aren't the only m-admissible pairs. We show that $(2, 5)$ is 11-admissible. For each $n \in \mathbb{N}$, let $S_{n,k} = \{n, n + 1, \ldots, n + k - 1\}$. Then

$$\{1, 3, 4\} \cup S_{6,41} = \{1, 6, 11, 16\} \cup \{3, 17, 31, 45\} \cup \{4, 18, 32, 46\} \cup S_{7,4} \cup S_{12,4} \cup S_{19,12} \cup S_{33,12}.$$

Alternatively, one can show that $(2,5)$ is 12-admissible via

$$\{1,3,4\} \cup S_{6,45} = \{1,4,7,10\} \cup \{3,6,9,12\} \cup \{8,13,18,23\} \cup \{11,24,37,50\} \cup S_{14,4} \cup S_{19,4} \cup S_{25,12} \cup S_{38,12}.$$

Extra Notes. By considering the numbers in the set mod 4, one can show that any m-admissible pair is congruent to $(1,2)$ or $(2,1) \pmod 4$. More precisely, for any set $S \subseteq \mathbb{Z}$ and for each $i = 0,1,2,3$, let $c_i(S)$ denote the number of elements of S that are congruent to $i \pmod 4$. One can show that if S can be partitioned into 4-term arithmetic progressions, then $c_0(S) \equiv c_2(S) \pmod 4$ and $c_1(S) \equiv c_3(S) \pmod 4$. As $\{1,2,\dots,4m+2\}$ contains $m+1$ integers congruent to $1 \pmod 4$ and $m+1$ integers congruent to $2 \pmod 4$, this gives an upper bound of at most $(m+1)^2 = m^2+2m+1$ m-admissible pairs.

On the other hand, for any m-admissible pair (i,j), both (i,j) and $(i+4,j+4)$ are also $(m+1)$-admissible. Thus, if $(2,5)$ is m_0-admissible for some m_0, then there are exactly $(m+1)^2$ m-admissible pairs for $m \geq 2m_0 - 1$. In particular, this holds for all $m \geq 2 \cdot 11 - 1 = 21$. This can be pushed further to $m \geq 11$ by checking that $(6,9)$ is 7-admissible via

$$S_{1,5} \cup \{7,8\} \cup S_{10,21} = \{1,4,7,10\} \cup \{2,5,8,11\} \cup \{3,12,21,30\} \cup S_{13,8} \cup S_{22,8}.$$

3: By boundedness, there exists a real number C such that $f(x) \geq 1 - Cx$ for all $x > 0$. By replacing C with $\max\{C,1\}$, we may assume that $C > 0$. Before we start with the main steps, note that by small induction, $a_n > 0$ for all $n \geq 1$.

Suppose for the sake of contradiction that $\sum_{n=1}^\infty a_n$ converges. Then there exists $N \geq 1$ such that $a_n < (2C)^{-1}$ for all $n \geq N$. Notice that for any $x < (2C)^{-1}$,

$$xf(x) \geq x(1-Cx) \implies \frac{1}{xf(x)} \leq \frac{1}{x(1-Cx)} = \frac{1}{x} + \frac{C}{1-Cx} \geq \frac{1}{x} + 2C.$$

Thus, we get $a_{n+1}^{-1} \leq a_n^{-1} + 2C$ for all $n \geq N$. By induction, we have $a_{N+k}^{-1} \leq a_N^{-1} + 2Ck$ for all $k \geq 0$. As a result,

$$\sum_{n=1}^\infty a_n \geq \sum_{k=1}^\infty a_{N+k} \geq \sum_{k=1}^\infty \frac{1}{a_N^{-1} + 2Ck},$$

which is a harmonic-type series and thus diverges. Contradiction!

4: Conjugation is often a powerful function in (non-abelian) group theory.

The equality $ba = ab^2$ can be rewritten as $a^{-1}ba = b^2$. By induction, one gets $a^{-n}ba^n = b^{2^n}$ for any integer $n \geq 0$. Since $a^{11} = 1$, this gives us $b = b^{2^{11}}$, or $b^{2^{11}-1} = 0$. Since p is the order of b, p divides $2^{11} - 1$.

For the converse, let p be a prime divisor of $2^{11} - 1$. In particular, p is odd. Consider the set $\mathrm{Sym}(\mathbb{Z}/p\mathbb{Z})$ of permutations on $\mathbb{Z}/p\mathbb{Z}$, which is a group. Let $G \leq \mathrm{Sym}(\mathbb{Z}/p\mathbb{Z})$ be the subgroup generated by the function $f(x) = 2x$ and $g(x) = x+1$; both are bijections. Since $2^{11} \equiv 1 \pmod p$, f has order 11. Clearly g has order p, since $p = 0$ in $\mathbb{Z}/p\mathbb{Z}$. Notice that $g \circ f = f \circ g \circ g$. Thus G is the desired group.

Finally, the prime divisors of $2^{11} - 1$ are $\boxed{\text{23 and 89.}}$

Note. A more general construction in the second part is known as the **semidirect product** of a group acting on another group. Then the case where the group being acted on is abelian, we can make the construction very concrete just like the one given above.

5: (The following solution assumes some knowledge of algebraic number theory. Is there a more elementary method?)

Since $\alpha^n + \alpha^{-n}$ is an integer, α is an algebraic integer. Furthermore, since $\alpha > 1$, α^n cannot be an integer. We claim that the only algebraic conjugates of α in $\overline{\mathbb{Q}} \subseteq \mathbb{C}$ are α itself and α^{-1}. Note that $\alpha \neq \alpha^{-1}$. Since α is an algebraic integer, the claim would then yield $\alpha + \alpha^{-1} \in \mathbb{Z}$. It remains to prove the claim.

Indeed, let β be an algebraic conjugate of α. Then β^n is an algebraic conjugate of α^n. Note that $\alpha^n + \alpha^{-n}$ and $\alpha^n \alpha^{-n} = 1$ are integers, so the only algebraic conjugates of α^n are α^n and α^{-n}. Thus, $\beta^n \in \{\alpha^n, \alpha^{-n}\}$. Similarly, $\beta^m \in \{\alpha^m, \alpha^{-m}\}$.

If $\beta^n = \alpha^n$ and $\beta^m = \alpha^m$, then by Euclidean algorithm and $\gcd(m, n) = 1$, it follows that $\beta = \alpha$. Similarly, if $\beta^n = \alpha^{-n}$ and $\beta^m = \alpha^{-m}$, then we get $\beta = \alpha^{-1}$. If $\beta^n = \alpha^n$ and $\beta^m = \alpha^{-m}$, then

$$1 = (\beta^n)^m (\beta^m)^{-n} = (\alpha^n)^m (\alpha^{-m})^{-n} = \alpha^{2mn}.$$

But α is a real number greater than 1; contradiction. Similarly, $\beta^n = \alpha^{-n}$ with $\beta^m = \alpha^m$ yields a contradiction. This proves the claim, as desired.

6: WLOG no three points are collinear. Consider the convex hull of the $(n+1)^2$ points. It is a convex k-gon with $(n+1)^2 - k$ interior points.

First assume that $k \leq 4n$. Then the convex k-gon is made up with $(k-2) + 2((n+1)^2 - k) = 2n^2 + 4n - k$ triangles. Thus, there is one with area at most

$$\frac{n^2}{2n^2 + 4n - k} \leq \frac{n^2}{2n^2} = \frac{1}{2}.$$

Suppose now $k > 4n$. The perimeter of the convex k-gon is at most $4n$, since the k-gon lies in the interior of a square of side length n. So we can find two consecutive edges with lengths a, b satisfying $\frac{a+b}{2} \leq \frac{4n}{k} < 1$. The area of this triangle with sides a, b is at most

$$\frac{ab}{2} \leq \frac{1}{2}\left(\frac{a+b}{2}\right)^2 < \frac{1}{2}.$$