Week 6: Mock Putnam 6

1: Prove that there exists infinitely many primes q that divide $p^2 - 2$ for some prime p.

2: Evaluate $\displaystyle\sum_{k=1}^{69} \frac{1}{k} \prod_{1 \le j \le 69, j \ne k} \frac{1}{k-j}$.

3: Let P be a real orthogonal matrix without eigenvalue 1. Let Q be obtained from P by replacing one of its rows or one of its columns by its negative. Show that Q has 1 as an eigenvalue.

4: Let $v_1, \ldots, v_n \in \mathbb{R}^m$ be unit vectors. Prove that there exist $\epsilon_1, \ldots, \epsilon_n, \delta_1, \ldots, \delta_n \in \{-1, 1\}$ such that

$$||\epsilon_1 v_1 + \cdots + \epsilon_n v_n|| \le \sqrt{n} \le \|\delta_1 v_1 + \cdots + \delta_n v_n\|,$$

where $\| \cdot \|$ denotes the Euclidean norm.

5: Evaluate

$$\int_0^1 \frac{\ln(\cos(\pi x/2))}{x}\, dx - \int_1^2 \frac{\ln(\sin(\pi x/2))}{x}\, dx.$$

6: Let $g : \mathbb{N} \to \mathbb{N}$ be a multiplicative function such that for any prime p and any $m \ge 1$, we have $g(p^m) = mp^{m-1}$. (Multiplicative means that $g(ab) = g(a)g(b)$ if $\gcd(a, b) = 1$.) Prove that there are infinitely many integers n such that $g(n) + 1 = g(n + 1)$.

Week 6: Sketch of proofs

1: We will use Dirichlet's theorem on arithmetic progressions.

Consider an arbitrary odd prime q that divide $n^2 - 2$ for some integer n. Note that $\gcd(n, q) = 1$ since $q \nmid 2$. By Dirichlet's theorem on arithmetic progressions, there exists a prime p such that $p \equiv n \pmod{q}$. Thus $p^2 - 2 \equiv n^2 - 2 \equiv 0 \pmod{q}$, yielding $q \mid p^2 - 2$. It remains to show that there exists infinitely many odd primes q that divides $n^2 - 2$ for some integer n.

Indeed, suppose for the sake of contradiction that only finitely many such q exists, say $q_1, \ldots, q_k$. Then $(q_1 q_2 \ldots q_k)^2 - 2$ is a large odd number that is not divisible by q_i for each $i \leq k$; contradiction.

Extra. Quadratic reciprocity tells us something stronger: for any $q \equiv \pm 1 \pmod{8}$, there exists an integer n such that q divides $n^2 - 2$.

2: Answer. $\dfrac{1}{69!}$.

Solution 1. For any positive integer N,

$$\begin{aligned}
\sum_{k=1}^{N} \frac{1}{k} \prod_{1 \leq j \leq N, j \neq k} \frac{1}{k-j} &= \sum_{k=1}^{N} \frac{1}{k} \prod_{1 \leq j < k} \frac{1}{k-j} \prod_{k < j \leq N} \frac{1}{k-j} \\
&= \sum_{k=1}^{N} \prod_{0 \leq j < k} \frac{1}{k-j} \prod_{k < j \leq N} \frac{-1}{j-k} \\
&= \sum_{k=1}^{N} \frac{(-1)^{N-k}}{k!(N-k)!} \\
&= \frac{1}{N!} \sum_{k=1}^{N} (-1)^{N-k} \binom{N}{k} \\
&= \frac{1}{N!} \left((1 + (-1))^N - (-1)^{N-0} \binom{N}{0} \right) \\
&= \frac{(-1)^{N+1}}{N!}.
\end{aligned}$$

Solution 2. Let L be the expression to be evaluated. Notice the polynomial equality

$$\begin{aligned}
L(X-1)(X-2)\ldots(X-69) &= \sum_{k=1}^{69} \frac{X-k}{k} \prod_{1 \leq j \leq 69, j \neq k} \frac{X-j}{k-j} \\
&= \sum_{k=1}^{69} \frac{X}{k} \prod_{1 \leq j \leq 69, j \neq k} \frac{X-j}{k-j} - \sum_{k=1}^{69} \prod_{1 \leq j \leq 69, j \neq k} \frac{X-j}{k-j}.
\end{aligned}$$

By Lagrange interpolation, the second summation term on the right hand side is 1. Thus plugging $X = 0$ yields $L \cdot (-1)^{69} 69! = -1$ and thus $L = \dfrac{1}{69!}$, as desired.

3: We start by proving a more general statement. Let P is a real orthogonal matrix. Suppose that 1 has multiplicity b as an eigenvalue of P, with $b = 0$ if 1 is not an eigenvalue of P. Then we claim that $\det(P) = (-1)^{n+b}$.

Let P is an $n \times n$ real orthogonal matrix. Let $f_P(X)$ be the characteristic polynomial of P. Since f_P has real coefficients, the complex roots come in complex conjugate pairs. Since P is orthogonal, the complex roots have absolute value 1. For $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $\lambda \neq \pm 1$, note that $\lambda + \lambda^{-1} \in (-2, 2)$. Thus we can write

$$f_P(X) = (X+1)^a (X-1)^b \prod_{i=1}^{k} (X^2 - c_i X + 1)$$

for some $c_1, c_2, \ldots, c_k \in (-2, 2)$ and $a, b \in \mathbb{N}_0$. In particular, $f_P(0) = (-1)^b$. On the other hand, $f_P(0) = (-1)^n \det(P)$, so we get $\det(P) = (-1)^{n+b}$, as desired.

We now go back to the main problem. Since P is orthogonal without eigenvalue 1, we have $\det(P) = (-1)^n$. By construction, Q is equal to either PD or DP for some diagonal matrix D with one entry equal to -1 and all other diagonal entries equal to 1. In particular, Q is orthogonal and $\det(Q) = -\det(P) \neq (-1)^n$. The inequality, with the above claim, implies that Q has 1 as an eigenvalue.

4: We present two solutions. Both of them rely on the parallelogram law: for any $v, w \in \mathbb{R}^m$,

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

Solution 1. We proceed by induction on n. The base case $n = 1$ is obvious, so we skip to the induction step.

Fix some unit vectors $v_1, v_2, \ldots, v_{n+1} \in \mathbb{R}^m$. By induction hypothesis, there exists $\epsilon_1, \ldots, \epsilon_n$, $\delta_1, \ldots, \delta_n \in \{-1, 1\}$ such that

$$\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\| \leq \sqrt{n} \leq \|\delta_1 v_1 + \cdots + \delta_n v_n\|.$$

For convenience, write $w_1 = \epsilon_1 v_1 + \cdots + \epsilon_n v_n$ and $w_2 = \delta_1 v_1 + \cdots + \delta_n v_n$.

Since $\|w_1\| \leq \sqrt{n}$, parallelogram law yields

$$\|w_1 + v_{n+1}\|^2 + \|w_1 - v_{n+1}\|^2 = 2\|w_1\|^2 + 2\|v_{n+1}\|^2 \leq 2n + 2 \cdot 1 = 2(n+1).$$

Thus either $\|w_1 + v_{n+1}\|^2 \leq n + 1$ or $\|w_1 - v_{n+1}\|^2 \leq n + 1$. In the former case, we take $\epsilon_{n+1} = 1$, and in the latter case, we take $\epsilon_{n+1} = -1$.

Similarly, since $\|w_2\| \leq \sqrt{n}$, parallelogram law yields

$$\|w_2 + v_{n+1}\|^2 + \|w_2 - v_{n+1}\|^2 = 2\|w_2\|^2 + 2\|v_{n+1}\|^2 \geq 2n + 2 \cdot 1 = 2(n+1).$$

Thus either $\|w_1 + v_{n+1}\|^2 \geq n + 1$ or $\|w_1 - v_{n+1}\|^2 \geq n + 1$. In the former case, we take $\delta_{n+1} = 1$, and in the latter case, we take $\delta_{n+1} = -1$.

Solution 2. This time, we just consider the averages of $\|\epsilon_1 v_1 + \ldots + \epsilon_n v_n\|^2$ across all $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$. It suffices to show that for any $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$,

$$\frac{1}{2^n} \sum_{\epsilon_1, \ldots, \epsilon_n \in \{-1,1\}} \|\epsilon_1 v_1 + \ldots + \epsilon_n v_n\|^2 = \sum_{k=1}^{n} \|v_k\|^2.$$

The right hand side is n if the v_i's are unit vectors. In that case, we are done.

Proceed by induction on n. The base case $n = 1$ is trivial, and the induction step follows from

$$\begin{aligned}
&\frac{1}{2^{n+1}} \sum_{\epsilon_1, \ldots, \epsilon_{n+1} \in \{-1,1\}} \|\epsilon_1 v_1 + \ldots + \epsilon_{n+1} v_{n+1}\|^2 \\
&= \frac{1}{2^{n+1}} \sum_{\epsilon_1, \ldots, \epsilon_n \in \{-1,1\}} \left(\|\epsilon_1 v_1 + \ldots + \epsilon_n v_n + v_{n+1}\|^2 + \|\epsilon_1 v_1 + \ldots + \epsilon_n v_n - v_{n+1}\|^2\right) \\
&= \frac{1}{2^{n+1}} \sum_{\epsilon_1, \ldots, \epsilon_n \in \{-1,1\}} 2\left(\|\epsilon_1 v_1 + \ldots + \epsilon_n v_n\|^2 + \|v_{n+1}\|^2\right) \\
&= \frac{1}{2^n} \sum_{\epsilon_1, \ldots, \epsilon_n \in \{-1,1\}} \|\epsilon_1 v_1 + \ldots + \epsilon_n v_n\|^2 + \|v_{n+1}\|^2.
\end{aligned}$$

5: Answer. $\frac{1}{2}(\ln 2)^2 - \ln \pi \ln 2$.

For each $n \geq 0$, let

$$L_n = \int_0^{1/2^n} \frac{\ln(\cos(\pi x)/2)}{x}\, dx - \int_1^2 \frac{\ln(\sin(\pi x/2^{n+1}))}{x}\, dx.$$

The goal is to evaluate L_0. First by the double angle formula, for any $n \geq 0$,

$$\begin{aligned}
L_n &= \int_0^{1/2^n} \frac{\ln(\cos(\pi x)/2)}{x}\, dx - \int_1^2 \frac{\ln 2}{x}\, dx - \int_1^2 \frac{\ln(\cos(\pi x/2^{n+2}))}{x}\, dx - \int_1^2 \frac{\ln(\sin(\pi x/2^{n+2}))}{x}\, dx \\
&= \int_0^{1/2^n} \frac{\ln(\cos(\pi x)/2)}{x}\, dx - (\ln 2)^2 - \int_{1/2^{n+1}}^{1/2^n} \frac{\ln(\cos(\pi x/2))}{x}\, dx - \int_1^2 \frac{\ln(\sin(\pi x/2^{n+2}))}{x}\, dx \\
&= \int_0^{1/2^{n+1}} \frac{\ln(\cos(\pi x)/2)}{x}\, dx - (\ln 2)^2 - \int_1^2 \frac{\ln(\sin(\pi x/2^{n+2}))}{x}\, dx \\
&= L_{n+1} - (\ln 2)^2.
\end{aligned}$$

Thus, by induction on n, we get $L_n = L_0 + n(\ln 2)^2$ for all $n \geq 0$. This implies

$$L_0 = \lim_{n\to\infty} L_n - n(\ln 2)^2 = \lim_{n\to\infty} \int_0^{1/2^n} \frac{\ln(\cos(\pi x/2))}{x}\, dx - n(\ln 2)^2 - \int_1^2 \frac{\ln(\sin(\pi x/2^{n+1}))}{x}\, dx.$$

We first approximate the first integral. The function $\ln(\cos(\pi x/2))/x$ is defined for $x \in (0, 1)$ since $\cos(\pi x/2) > 0$. Furthermore, $\ln(\cos 0) = 0$, so we can apply L'Hopital's rule and get

$$\lim_{x\to 0^+} \frac{\ln(\cos(\pi x)/2)}{x} = \lim_{x\to 0^+} \frac{-\pi/2 \cdot \sin(\pi x/2)}{\cos(\pi x/2)} = -\frac{\pi}{2} \lim_{x\to 0^+} \tan(\pi x/2) = 0.$$

Thus, we have

$$\lim_{n\to\infty}\int_0^{1/2^n}\frac{\ln(\cos(\pi x)/2)}{x}\,dx = 0.$$

We now approximate the second integral. Notice that for any $x \in [1,2]$, we have

$$\ln(\sin(\pi x/2^n)) - \ln(\pi x/2^n) = \ln\left(\frac{\sin(\pi x/2^n)}{\pi x/2^n}\right) \xrightarrow{n\to\infty} \ln 1 = 0.$$

Since $[1,2]$ is compact, the above convergence is also uniform, and thus

$$\int_1^2 \frac{\ln(\sin(\pi x/2^{n+1})) - \ln(\pi x/2^{n+1})}{x}\,dx \xrightarrow{n\to\infty} 0.$$

Finally, for any n,

$$\begin{aligned}\int_1^2 \frac{\ln(\pi x/2^{n+1})}{x}\,dx &= \int_1^2 \frac{\ln\pi}{x}\,dx + \int_1^2 \frac{\ln x}{x}\,dx - \int_1^2 \frac{(n+1)\ln 2}{x}\,dx\\ &= \ln\pi\ln 2 + \int_0^{\ln 2} y\,dy - (n+1)(\ln 2)^2\\ &= \ln\pi\ln 2 - \frac{1}{2}(\ln 2)^2 - n(\ln 2)^2.\end{aligned}$$

Thus $L_0 = -(\ln\pi\ln 2 - \frac{1}{2}(\ln 2)^2) = \frac{1}{2}(\ln 2)^2 - \ln\pi\ln 2$.

6: We start with two important observations. First, note that $g(p) = 1$ for all primes p. Since g is multiplicative, we have $g(p_1 p_2 \dots p_m) = 1$ for all m distinct primes $p_1, p_2, \dots, p_m$. That is, we have $g(x) = 1$ whenever x is a squarefree positive integer. Second, notice that $g(27) = g(169) + 1 = 27$.

The main idea of this solution is that we want to find infinitely many pairs of squarefree positive integers (a,b) such that $\gcd(a,27) = \gcd(b,169) = 1$ and $27a = 169b + 1$. Indeed, the first condition and squarefree-ness yield $g(27a) = g(27)g(a) = 27$ and $g(169b) = g(169)g(b) = 26$. Then the last equality yields $g(169b+1) = 27 = g(169b) + 1$. It remains to show that there exists infinitely many pairs of squarefree positive integers (a,b) such that $\gcd(a,3) = \gcd(b,13) = 1$ and $27a = 169b + 1$.

By extended Euclidean algorithm (or guessing), one finds $27 \cdot 25 = 169 \cdot 4 - 1$. Thus we can parametrize the pairs (a,b) such that $27a = 169b+1$ by $(a,b) = (169n - 25, 27n - 4)$ for some positive integer n. To remove some technicalities, we consider pairs (a,b) of form $(3 \cdot 13^3 n - 25, 3^4 \cdot 13 - 4)$ instead, so that $\gcd(a,3) = \gcd(b,13) = 1$ is automatic. Let S be the set of $n \in \mathbb{N}$ such that $3 \cdot 13^3 n - 25$ and $3^4 \cdot 13n - 4$ are squarefree. Note that $\gcd(3 \cdot 13^3 n - 25, 3^4 \cdot 13n - 4) = 1$, so this is equivalent to saying that $P(n)$ is squarefree, where $P \in \mathbb{Z}[X]$ is the polynomial

$$P(X) = (3 \cdot 13^3 X - 25)(3^4 \cdot 13X - 4).$$

The goal now reduces to prove that S is infinite. In fact, we will prove more:

$$\liminf_{X\to\infty}\frac{|S \cap [1,X]|}{X} > 0.$$

For each $n \in (\mathbb{N} \cap [1, X]) \setminus S$, by definition, there exists a prime p such that $p^2 \mid P(n)$. However, since $P(n) = (3 \cdot 13^3 n - 25)(3^4 \cdot 13n - 4)$ with $\gcd(3 \cdot 13^3 n - 25, 3^4 \cdot 13n - 4) = 1$, this means that p^2 divides either $3 \cdot 13^3 n - 25$ or $3^4 \cdot 13n - 4$. That is, we have

$$n \equiv \frac{25}{3 \cdot 13^3} \pmod{p^2} \quad \text{or } n \equiv \frac{4}{3^4 \cdot 13} \pmod{p^2}.$$

The number of such n in the interval $[1, X]$ is at most $2(\lfloor X/p^2 \rfloor + 1) = 2X/p^2 + O(1)$. Furthermore, it is zero if $p^2 > \max\{3 \cdot 13^3 X - 25, 3^4 \cdot 13X - 4\}$. In particular, this is true if say $p > C\sqrt{X}$, where $C = 100$. The number is also zero if p equals 3 or 13. As a result, by union bound,

$$|(\mathbb{N} \cap [1, X]) \setminus S| \leq \sum_{\substack{p < C\sqrt{X} \\ p \neq 3,13}} \left(\frac{2}{p^2} X + O(1) \right) \leq \sum_{\substack{p \text{ prime} \\ p \neq 3,13}} \frac{2}{p^2} X + O(\sqrt{X}).$$

On the other hand,

$$\sum_{\substack{p \text{ prime} \\ p \neq 3,13}} \frac{1}{p^2} \leq \frac{1}{4} + \sum_{p=5}^{\infty} \frac{1}{p^2} = \frac{1}{4} + \frac{\pi^2}{6} - 1 - \frac{1}{9} - \frac{1}{16} = \frac{\pi^2}{6} - \frac{169}{144}.$$

This implies that

$$\liminf_{X \to \infty} \frac{|S \cap [1, X]|}{X} \geq 1 - 2\left(\frac{\pi^2}{6} - \frac{169}{144} \right) = \frac{241}{72} - \frac{\pi^2}{3} > \frac{10 - \pi^2}{3} > 0.$$

Extra. Just with a slightly more careful bound, we can get

$$\liminf_{X \to \infty} \frac{|S \cap [1, X]|}{X} \geq 1 - 2\left(\frac{1}{4} + \sum_{n=2}^{\infty} \frac{1}{(2n+1)^2} \right) \geq 1 - 2\left(\frac{1}{4} + \frac{\pi^2}{8} - 1 - \frac{1}{9} \right) > \frac{1}{4}.$$

Note that this proof also shows that

$$\liminf_{X \to \infty} \frac{\#\{n \in \mathbb{N} : n < X, g(n+1) = g(n) + 1\}}{X} > 0.$$