Solutions to the Combinatorics Problems

1: Find the number of words of length n on the alphabet $\{0, 1\}$ with exactly m blocks of the form 01.

Solution: There are $n-1$ locations between the digits in such a word. Let us call a location at which the digits switch (either from 0 to 1 or from 1 to 0) a switch-location. For a word of the required form which starts and ends with a 1, there must be $2m$ switch-locations (with every second switch-location giving a 01-block) so there are $\binom{n-1}{2m}$ such words. For a word of the required form which starts with a 1 and ends with a 0, there must be $2m+1$ switch-locations, so there are $\binom{n-1}{2m+1}$ such words. A word that starts with 0 and ends with 1 must have $2m-1$ switch-locations, so there are $\binom{n-1}{2m-1}$ such words, and a word that starts and ends with 0 must have $2m$ swith-locations, so there are $\binom{n-1}{2m}$ such words. Altogether there are $\binom{n-1}{2m} + \binom{n-1}{2m+1} + \binom{n-1}{2m-1} + \binom{n-1}{2m} = \binom{n}{2m+1} + \binom{n}{2m} = \binom{n+1}{2m+1}$ such words.

Alternatively, a nice trick is to note that if we append a 1 to the beginning and a 0 to the end of a word of the required form, then the new word will be of length $n+2$ and will still have m 01-blocks; there will be $n+1$ locations between the digits in the word, and $2m+1$ of these locations will be switch-locations, so there are $\binom{n+1}{2m+1}$ such words.

2: Find the number of words of length n on the alphabet $\{0, 1, 2, 3\}$ with an even number of zeros.

Solution: Let a_n be the number of words of length n with an even number of 0's, and let b_n be the number of words of length n with an odd number of 0's. Note that any word of length $n+1$ with an even number of 0's can be obtained either by appending a 1, 2 or 3 to the end of a word of length n with an even number of 0's, or by appending a 0 to the end of a sequence of length n with an odd number of 0's, and so we have the recurrence relation $a_{n+1} = 3a_n + b_n$. Similarly, we have and $b_{n+1} = a_n + 3b_n$. The first few values of a_n and b_n are listed below.

n	1	2	3	4	$\cdots$
a_n	3	10	36	136	$\cdots$
b_n	1	6	28	120	$\cdots$

We can combine the recurrence formulas for $\{a_n\}$ and $\{b_n\}$ to get a single recurrence formula for $\{a_n\}$ as follows.

$$a_{n+2} = 3a_{n+1} + b_{n+1} = 3a_{n+1} + (a_n + 3b_n) = 3a_{n+1} - 8a_n + (9a_n + 3b_n) = 3a_{n+1} - 8a_n + 3a_{n+1} = 6a_{n+1} - 8a_n$$

To solve this, we solve its characteristic equation $\lambda^2 - 6\lambda + 8 = 0$ to get $\lambda = 2, 4$, so the formula for a_n is of the form $a_n = A \cdot 2^n + B \cdot 4^n$. Put in $n = 1$ and $n = 2$ to get $2A + 4B = 3$ and $4A + 16B = 10$. Solve these to get $A = B = \frac{1}{2}$, and so we have $a_n = \frac{1}{2}\left(2^n + 4^n\right)$.

3: Find the number of words of length n on the alphabet $\{0, 1, 2\}$ such that neighbours differ by at most 1.

Solution: Let a_n be the number of such words that end with 0, let b_n be the number of such words that end with 1, let c_n be the number of such words that end with 2, and let x_n be the total number of such words, so $x_n = a_n + b_n + c_n$. By interchanging 0's and 2's we obtain a bijection between the set of words of the required form that end with 0 with the set of such words that end with 2, and so we have $a_n = c_n$ and $x_n = 2a_n + b_n$. Note that $a_1 = b_1 = 1$, and we have the recursion $a_{n+1} = a_n + b_n$ and $b_{n+1} = a_n + b_n + c_n = 2a_n = x_n$. The first few values are listed below.

n	1	2	3	4	5	$\cdots$
a_n	1	2	5	12	29	$\cdots$
b_n	1	3	7	17	41	$\cdots$
x_n	3	7	17	41	99	$\cdots$

We can combine these paired recurrence formulas for $\{a_n\}$ and $\{b_n\}$ to get a single one for $\{b_n\}$ as follows.

$$b_{n+2} = 2a_{n+1} + b_{n+1} = 2(a_n + b_n) + b_{n+1} = (2a_n + b_n) + b_n + b_{n+1} = b_{n+1} + b_n + b_{n+1} = 2b_{n+1} + b_n \,.$$

To solve this, we solve its characteristic equation $\lambda^2 - 2\lambda - 1 = 0$ to get $\lambda = 1 \pm \sqrt{2}$, so the formula for b_n is of the form $b_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n$ $(*)$. Extend the sequence $\{b_n\}$ to include $b_0 = 1$ (so the recurrence relation is still satisfied), then put $n = 0$ and $n = 1$ into equation $(*)$ to get $A + B = 1$ (1) and $A(1+\sqrt{2}) + B(1-\sqrt{2}) = 1$ (2).

Solve equations (1) and (2) to get $A = B = \frac{1}{2}$, and so $b_n = \frac{1}{2}\big((1+\sqrt{2})^n + (1-\sqrt{2})^n\big)$. Thus the number of words of the required form is $x_n = b_{n+1} = \frac{1}{2}\big((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\big)$.

4: Find the number of words on the alphabet $\{0,1,2\}$ with no neighbouring zeros.

Solution: This is similar to problem 3. The answer is

$$\tfrac{1}{6}\left((3+2\sqrt{3})(1+\sqrt{3})^n + (3-2\sqrt{3})(1-\sqrt{3})^n\right) .$$

5: Find the number of subsets of $\{1,2,\cdots,n\}$ which do not contain two successive numbers.

Solution: Given a subset $A \subset \{1,2,\cdots,n\}$ we associate the word $e_1e_2\cdots e_3$ on $\{0,1\}$ given by $e_k = \begin{cases} 1 \text{ if } k \in A \\ 0 \text{ if } k \notin A \end{cases}$. Note that A contains two successive numbers if and only if the word has a block of the form 11. Thus the required number of subsets is equal to the number of words of length n on $\{0,1\}$ with no 11-blocks. Let a_n be the number of such words that end with 0, let b_n be the number of such words that end with 1, and let x_n be the total number of such words so $x_n = a_n + b_n$. Then $a_1 = b_1 = 1$ and we have the recurrence formulas $a_{n+1} = a_n + b_n$ and $b_{n+1} = a_n$, so $x_n = a_{n+1}$. We combine these to get $a_{n+2} = a_{n+1} + b_{n+1} = a_{n+1} + a_n$, so we see that $a_n = f_{n+1}$ and so $x_n = f_{n+2}$, where f_n denotes the n^{th} Fibonacci number. Solving the recursion formula for the Fibonacci numbers gives $f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$, so $x_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right)$.

Alternatively, if A contain n, then it does not contain $n-1$ and the rest of A is a subset of $\{1,2,\ldots,n-2\}$ with no successive numbers. If A does not contain n, then A is a subset of $\{1,2,\ldots,n-1\}$ with no successive numbers. Hence $a_n = a_{n-1} + a_{n-2}$.

6: Find the number of ways to choose two disjoint nonempty subsets from the set $\{1,2,\cdots,n\}$.

Solution: To a given ordered pair (A,B) of disjoint subsets $A, B \subset \{1,2,\cdots,n\}$, we associate the word $e_1e_2\cdots e_n$ on $\{0,1,2\}$ given by

$$e_k = \begin{cases} 0 \text{ if } k \notin A \cap B \\ 1 \text{ if } k \in A \\ 2 \text{ if } k \in B \end{cases} .$$

We have $A = \emptyset$ if and only if the associated word is a word on $\{0,1\}$, and $B = \emptyset$ if and only if the associated word is a word on $\{0,2\}$. Of the 3^n words of length n, there are 2^n which are words on $\{0,1\}$ and 2^n which are words on $\{0,2\}$, and only 1 which is a word on $\{0\}$. Thus the number of ordered pairs of disjoint subsets of $\{1,2,\cdots,n\}$ is equal to $3^n - 2\cdot 2^n + 1$, and so the number of unordered pairs of disjoint subsets is $\frac{1}{2}(3^n+1) - 2^n$.

Alternatively, for each $k = 1,\ldots,n-1$, there are $\binom{n}{k}$ ways to choose the first subset to have size k and then $2^{n-k} - 1$ ways to choose the second subset to be disjoint. Summing over k and dividing by 2 for overcounting gives the desired result.

7: Find the number of surjective maps from the set $\{1,2,3,4,5,6\}$ to the set $\{1,2,3,4\}$.

Solution: Note first that there are n^k maps from any set of k elements to any set of n elements, (since there are n choices for the image of each of the k elements), but some of these maps are not surjective. Let A be a set with k elements and let $B = \{b_1, b_2, \cdots, b_n\}$ be a set with n elements. For each $i = 1, \cdots, n$, let $B_i = B \setminus \{b_i\}$. Note that a map from A to B is *not* surjective when its image lies in one of the subsets B_i. Let S_i be the set of maps from A to B_i. Note that for $i < j$, $S_i \cap S_j$ is the set of maps from A to $B_i \cap B_j = B \setminus \{i,j\}$, and for $i < j < k$, $S_i \cap S_j \cap S_k$ is the set of maps from A to $B \setminus \{i,j,k\}$, and so on. By the remark made in our first sentence, we have $|S_i| = (n-1)^k$, $|S_i \cap S_j| = (n-2)^k$, $|S_i \cap S_j \cap S_k| = (n-3)^k$ and so on. By the Principle of Inclusion and Exclusion, the total number of non-surjective maps from A to B is

$$\begin{aligned} |S_1 \cup \cdots \cup S_n| &= \sum_i |S_i| - \sum_{i<j} |S_i \cap S_j| + \sum_{i<j<k} |S_i \cap S_j \cap S_k| - \cdots \\ &= \binom{n}{1}(n-1)^k - \binom{n}{2}(n-2)^k + \binom{n}{3}(n-3)^k - \cdots \pm \binom{n}{n-1}(1)^k \end{aligned}$$

Thus the number of surjective maps from A to B is

$$n^k - \binom{n}{1}(n-1)^k + \binom{n}{2}(n-2)^k - \binom{n}{3}(n-3)^k + \cdots \pm \binom{n}{n-1}(1)^k$$

In particular, when $k=6$ and $n=4$, there are $4^6-4\cdot 3^6+6\cdot 2^6-4\cdot 1^6=4096-2916+384-4=1560$.

8: Find the number of permutations of order 6 in the group of all permutations of $\{1,2,\cdots,8\}$.

Solution: Given distinct elements $a_1,a_2,\cdots,a_l\in\{1,2,\cdots,n\}$, we write $(a_1,a_2,\cdots,a_l)$ for the permutation σ of $\{1,2,\cdots,n\}$ defined by $\sigma(a_1)=a_2$, $\sigma(a_2)=a_3,\cdots,\sigma(a_{l-1})=a_l$ and $\sigma(a_l)=a_1$ and $\sigma(k)=k$ if $k\neq a_i$ for any i. Such a permutation is called a **cycle** of length l. Two cycles $(a_1,\cdots,a_k)$ and $(b_1,\cdots,b_l)$ are called **disjoint** when $a_i\neq b_j$ for any i,j. The following facts are well known and not difficult to prove.

1. Every permutation of $\{1,2,\cdots,n\}$ is a product of disjoint cycles, and the product is unique up to the order of the cycles and the cyclic ordering of the elements in each cycle.
2. The order of a product of disjoint cycles is the least common multiples of their lengths.

We illustrate how to count the number of permutations of $\{1,2,\cdots,n\}$ which are equal to a product of disjoint cycles of specified lengths, by finding the number of permutations of $\{1,2,\cdots,26\}$ of the form

$$(abcdef)(ghij)(klmn)(opq)(rst)(uvw)\,.$$

There are $\binom{26}{6}$ ways to choose the 6 unordered elements a,b,c,d,e,f. We can take a to be the smallest of these, then there are 5! ways to choose the remaining 5 ordered elements b,c,d,e,f. Next there are $\binom{20}{8}$ ways to choose the next 8 unordered elements g,h,i,j,k,l,m,n. We take g to be the smallest of these 8, then there are $7\cdot 6\cdot 5$ ways to choose the ordered elements h,i,j, then we take k to be the smallest of the 4 elements k,l,m,n, and then there are $3\cdot 2\cdot 1$ ways to choose the ordered elements l,m,n. Finally, there are $\binom{12}{9}$ ways to choose the unordered elements o,p,q,r,s,t,u,v,w, we take o to be the smallest, there are $8\cdot 7$ choices for p,q, we take r to be the smallest of the 6 elements r,s,t,u,v,w, there are $5\cdot 4$ choices for s,t, then we take u to be the smallest of u,v,w and there are $2\cdot 1$ choices for v,w. Thus the total number of permutations of the above form is equal to

$$\binom{26}{6}5\cdot 4\cdot 3\cdot 2\cdot 1\binom{20}{8}7\cdot 6\cdot 5\cdot 3\cdot 2\cdot 1\binom{12}{9}8\cdot 7\cdot 5\cdot 4\cdot 2\cdot 1\,.$$

Using this counting method, we now count the total number of permutations of $\{1,2,\cdots,8\}$ of order 6. We make a table showing all possible forms for such permutations, and the number of permutations of each form.

form	no. of elements
$(abc)(de)$	$\binom{8}{3}\cdot 2\cdot\binom{5}{2}=1120$
$(abc)(de)(fg)$	$\binom{8}{3}\cdot 2\cdot\binom{5}{4}\cdot 3=1680$
$(abc)(def)(gh)$	$\binom{8}{6}\cdot 5\cdot 4\cdot 2=1120$
$(abcdef)$	$\binom{8}{6}\cdot 5!=3360$
$(abcdef)(gh)$	$\binom{8}{6}\cdot 5!=3360$

Thus the total number is $1120+1680+1120+3360+3360=10640$.

9: (a) Into how many regions do n great circles divide the surface of a sphere, given that no three of the great circles intersect at a point?

Solution: Each of the $\binom{n}{2}$ pairs of great circles intersect in two points, so the total number of points (or vertices) is $V = 2\binom{n}{2} = n(n-1)$. Each of the n great circles meets each of the other $(n-1)$ great circles at two points, so there are $2(n-1)$ points along each great circle, so each great circle is divided into $2(n-1)$ arcs (or edges), and the total number of edges is $E = 2n(n-1) = 2V$. The Euler characteristic of the sphere is $\chi = 2$ so we have $V - E + F = 2$ where F is the required number of regions (or faces). Thus $F = E - V + 2 = V + 2 = n^2 - n + 2$.

(b) Into how many regions do n spheres divide space, given that any two of the spheres intersect along a circle, no three intersect along a circle, and no four intersect at a point?

Solution: Let a_n denote the required number of regions. Note that $a_1 = 2$ and $a_2 = 4$. When we add an $(n+1)^{\text{st}}$ sphere, it intersects the other n spheres along n circles, each pair of which intersect at two points and no 3 of which intersect at a point. By the proof of part (a) (with every occurrence of the word "great" removed) these n circles divide the $(n+1)^{\text{st}}$ sphere into $n^2 - n + 2$ regions. Each of the regions corresponds to a subdivision of a region in space into two parts, so we obtain the following recursion formula:

$$a_{n+1} = a_n + (n^2 - n + 2)\,.$$

Thus we have

$$a_n = 2 + 2 + 4 + \cdots + ((n-1)^2 - (n-1) + 2) = \sum_{k=0}^{n-1} k^2 - k + 2\,.$$

Evaluate this sum to get $a_n = \dfrac{n(n^2 - 3n + 8)}{3}$.

10: Find the number of paths in the set $\left\{(x,y) \in \mathbf{Z}^2 \middle| 0 \le y \le x\right\}$ which move always to the right or upwards from the point $(0,0)$ to the point (n,n).

Solution: First we solve the easier problem of finding the number of paths in $\mathbf{Z}^2$ which move always to the right and upwards from the point (a,b) to the point $(a+k, b+l)$. Such a path consists of $k+l$ steps with k of the steps to the right and l of the steps upwards, so it corresponds in a natural way to a word of length $k+l$ on $\{r,u\}$ with k r's and l u's (with each r indicating a step to the right and each u indicating a step upwards). There are $\binom{k+l}{k}$ such words, and hence the same number of such paths.

Now we return to the given problem. From the previous paragraph we know that there are $\binom{2n}{n}$ paths from $(0,0)$ to (n,n) (moving upwards and to the right). Let us call such a path **good** if it remains below or touches the line $y = x$, and let us call such a path **bad** if it crosses the line $y = x$, that is if it touches the line $y = x + 1$. We must count the number of good paths.

There is a lovely trick which helps us to count the number of bad paths. Given a bad path from $(0,0)$ to (n,n) we associate a path from $(-1,1)$ to (n,n) as follows: find the first point p where the bad path touches the line $y = x + 1$ and reflect the initial portion of the bad path (the portion from $(0,0)$ to p) in the line $y = x + 1$ to obtain a path from $(-1,1)$ to (n,n). Notice that we can recover the given bad path from the resulting path by performing the same operation, and so this gives a bijective correspondence between the set of bad paths from $(0,0)$ to (n,n) and the set of all paths from $(-1,1)$ to (n,n). By the result of the first paragraph there are $\binom{2n}{n+1}$ such paths. Thus the total number of good paths from $(0,0)$ to (n,n) is $\binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$.

11: $2n$ distinct points lie on a circle. In how many ways can the points be paired so that when all pairs are joined by line segments, then resulting n line segments are disjoint.

Solution: Let c_n denote the number of ways that $2n$ points on a circle can be paired so that the various line segments joining the pairs do not cross. Let a_1 be one of the $2n$ points, and let $a_2, a_3, \cdots, a_{2n}$ be the rest of the points in order around the circle (say clockwise). Note that a_1 cannot be paired with a_k for k odd, since if it were then we would have an odd number of points $a_2, a_3, \cdots, a_{k-1}$ between a_1 and a_k, one of which would have to be paired with a point on the other side of the line segment $a_1 a_k$. Thus a_1 can only be paired with $a_2, a_4, \cdots, a_{2n}$. When a_1 is paired with a_{2k}, the $2(k-1)$ points $a_2, a_3, \cdots, a_{2k-1}$ points must be paired amongst themselves and there are c_{k-1} ways to do this, and the $2(n-k)$ points $a_{k+1}, a_{k+2}, \cdots a_{2n}$ must be paired amongst themselves and there are c_{n-k} ways to do this. Thus, setting $c_0 = 1$, we have the following recurrence relation for $\{c_n\}$:

$$c_n = c_0 c_{n-1} + c_1 c_{n-2} + c_2 c_{n-3} + \cdots + c_{n-1} c_0 \,.$$

The first few values are as follows

n	0	1	2	3	4	$\cdots$
c_n	1	1	2	5	14	$\cdots$

To solve this recurrence relation, let $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$. Then

$$f(x)^2 = (c_0 c_0) + (c_0 c_1 + c_1 c_0)\, x + (c_0 c_2 + c_1 c_1 + c_2 c_0)\, x^2 + \cdots = c_1 + c_2 x + c_3 x^2 + \cdots$$

and so we have $x\, f(x)^2 = f(x) - 1$. By the quadratic formula, we have $f(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$. In order to have $\lim\limits_{x \to 0} f(x) = c_0$, we must use the negative sign, so $f(x) = \dfrac{1 - \sqrt{1-4x}}{2x}$. Using the binomial expansion $(1-x)^{1/2} = 1 - \sum\limits_{n=1}^{\infty} \frac{2}{n} \frac{(2n-2)!}{\left(2^n (n-1)!\right)^2}$ we obtain

$$f(x) = \frac{1}{2x}\left(1 - \sqrt{1-4x}\right) = \frac{1}{2x} \sum_{n=1}^{\infty} \frac{2}{n} \binom{2n-2}{n-1} x^n = \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n \,.$$

Thus we obtain $c_n = \frac{1}{n+1}\binom{2n}{n}$. These numbers c_n are called the **Catalan numbers**.

It seems a most remarkable coincidence that this problem has the same answer as the previous problem. There is a wonderful way to see why this relationship holds. Given a pairing of the $2n$ points $a_1, a_2, \cdots, a_{2n}$ on the circle, connect them by nonintersecting line segments then form a word $e_1 e_2 \cdots e_{2n}$ on $\{r, u\}$ as follows. Begin at a_1 and set $e_1 = r$, then move clockwise around the circle visiting the vertices $a_2, a_3, \cdots$. When we arrive at the vertex a_k, which is an end point of some line segment, set $e_k = r$ if it is the first time that we have visited this line segment, and set $e_k = u$ if it is the second time we have visited the line segment. Convince yourself that this word corresponds to a good path from $(0,0)$ to (n,n) and that the correspondence is bijective.

12: In how many ways can you triangulate a convex n-gon?

Solution: Let t_n denote the number of such triangulations. Note that $t_3 = 1$. Label the vertices of a given convex n-gon by $a_1, a_2, \cdots, a_n$ in order around the edge (say clockwise). Consider the edge $a_1 a_2$. It must be an edge of a triangle in any triangulation. If $a_1 a_2 a_3$ is a triangle in some triangulation, then the $(n-1)$-gon $a_1 a_3 a_4 \cdots a_n$ will be triangulated (by that same triangulation); if $a_1 a_2 a_n$ is a triangle in some triangulation then the $(n-1)$-gon $a_2 a_3 \cdots a_n$ will also be triangulated; and if $a_1 a_2 a_k$ is a triangle in some triangulation where $3 < k < n$, then both the $(k-1)$-gon $a_2 a_3 \cdots a_k$ and also the $(n-k+2)$-gon $a_k a_{k+1} \cdots a_n$ will be triangulated. Thus, setting $t_2 = 1$, we obtain the following recurrence formula for $\{t_n\}$:

$$t_n = t_2 t_{n-1} + t_3 t_{n-2} + t_4 t_{n-3} + \cdots + t_{n-1} t_2 \,.$$

This is the same recursion formula satisfied by the Catalan numbers, but with the indices shifted by 2, so we have $t_n = c_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$.