Combinatorics of Electrical Networks

David G. Wagner
Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1
dgwagner@math.uwaterloo.ca

I. Matrix-Tree Theorems.
II. Kirchhoff's Formula.
III. The Half-Plane Property and Rayleigh Monotonicity.
IV. Random Walks and Electrical Networks.

A series of lectures prepared for the
Undergraduate Summer Research Assistants
in the
Department of Combinatorics and Optimization
at the
University of Waterloo
in June and July of 2009.

I. Matrix-Tree Theorems.

Let $G = (V, E)$ be a finite graph which may contain loops or multiple edges, with $|V| = n$ vertices and $|E| = m$ edges. We denote by $\varkappa(G)$ the number of spanning trees of G, sometimes called the *complexity* of G. In this section we derive an easily computable formula for $\varkappa(G)$, and similar formulas for related generating series.

If G is not connected then $\varkappa(G) = 0$, so we can assume that G is connected from now on. If G' is obtained from G by removing all the loops of G, then $\varkappa(G') = \varkappa(G)$, since a loop can never occur in a spanning tree. Thus, we may assume that G contains no loops as well. Multiple edges, however, do remain a possibility.

The quantity $\varkappa(G)$ can be computed recursively, using the formula

$$\varkappa(G) = \varkappa(G \smallsetminus e) + \varkappa(G/e),$$

in which $G \smallsetminus e$ is the graph obtained from G by deleting the edge e and G/e is the graph obtained from G by contracting the edge e (and removing any loops produced). [Proof of this formula is a simple case analysis: $\varkappa(G \smallsetminus e)$ counts the spanning trees of G that do not contain the edge e, and $\varkappa(G/e)$ counts the spanning trees of G that do contain the edge e.] Furthermore, if G and H are connected graphs which intersect in exactly one vertex (and no edges) then

$$\varkappa(G \cup H) = \varkappa(G) \cdot \varkappa(H),$$

as is also easily seen. See Figure 1 for an example computation using this method. (For each graph in the figure, an edge which is deleted/contracted is marked with an asterisk.)

This recursion shows that the function $\varkappa$ is recursively enumerable, but the resulting algorithm in general requires on the order of $2^{|E|}$ arithmetic operations, and so it is not suitable for large computations.

Fortunately, there is an easily computable and completely general formula for $\varkappa(G)$, which we now derive. To state the formula we need to define some matrices. The *adjacency matrix* of G is the square matrix $A = A(G)$ indexed by $V \times V$, which has as its entries: $A_{vv} = 0$ for all $v \in V$, and if $v \neq w$ in V then A_{vw} is the number of edges of G which have vertices v and w at their ends. The degree $\deg_G(v)$ of a vertex $v \in V$ of G is the number of edges of G which are incident with v. The *degree matrix* of G is the diagonal V-by-V matrix $\Delta = \Delta(G)$

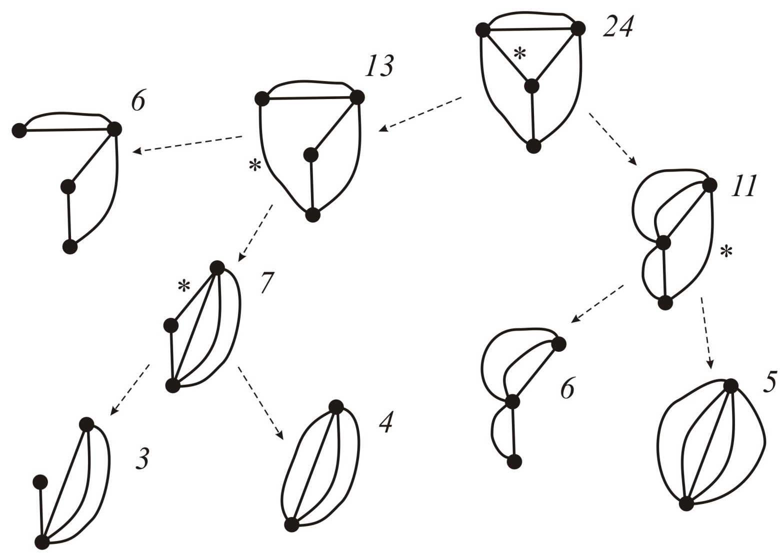

FIGURE 1. Computing $\varkappa(G)$ by deletion/contraction.

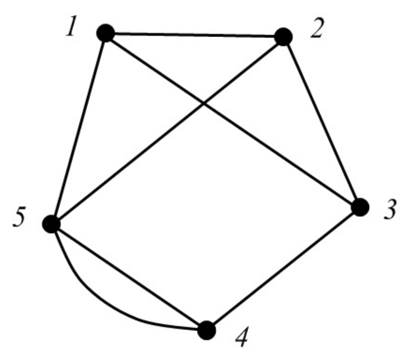

FIGURE 2

such that $\Delta_{vv} = \deg_G(v)$ for all $v \in V$, and $\Delta_{vw} = 0$ if $v \neq w$. Finally, the *Laplacian matrix* of G is defined to be $L(G) = \Delta(G) - A(G)$.

For example, the Laplacian matrix of the graph pictured in Figure 2 is

$$\begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -2 \\ -1 & -1 & 0 & -2 & 4 \end{bmatrix}$$

One more piece of notation is required. If M is a matrix and i is a row–index for M and j is a column–index for M, let $M(i|j)$ denote the submatrix of M obtained by deleting row i and column j from M.

Theorem 1 (The Matrix-Tree Theorem)**.** *Let $G = (V, E)$ be a graph, and let $L = L(G)$ be the Laplacian matrix of G. Then for any $v \in V$,*

$$\varkappa(G) = \det L(v|v).$$

Note that this determinant can be evaluated using a number of arithmetic operations that is bounded by a polynomial in $|V|$ (in fact, by $O(|V|^{2.54})$, I think). The Laplacian matrix $L(G)$ is also easy to construct from G. Thus, the Matrix-Tree Theorem is much more effective than the deletion/contraction recursion for computing $\varkappa(G)$.

One could prove Theorem 1 by induction on the number of edges, by showing that the right-hand side of the formula satisfies the same deletion/contraction recursion as does $\varkappa(G)$. The initial conditions forming the base case of the induction are easily checked. However, there is a more informative proof which also yields various generalizations of this Matrix-Tree Theorem.

We begin by expressing the Laplacian matrix of a graph in a different form. Consider a graph $G = (V, E)$, and orient each edge of G arbitrarily by putting an arrow on it pointing towards one of its two ends. The *signed incidence matrix* of G (with respect to this orientation) is the V-by-E indexed matrix D with entries

$$D_{ve} = \begin{cases} +1 & \text{if } e \text{ points in to } v \text{ but not out,} \\ -1 & \text{if } e \text{ points out of } v \text{ but not in,} \\ 0 & \text{otherwise.} \end{cases}$$

Figure 3 shows the graph of Figure 2 with the edges oriented arbitrarily and labelled with the letters from a to h. Relative to this orientation the signed incidence matrix of the graph is

$$\begin{bmatrix} -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \end{bmatrix}$$

(The rows are indexed by 1...5 and the columns by $a...h$.)

For any matrix M we denote the conjugate transpose of M by $M^\dagger$.

Lemma 2. *Let $G = (V, E)$ be a graph, orient G arbitrarily, and let D be the corresponding signed incidence matrix. Then $DD^\dagger = L(G)$.*

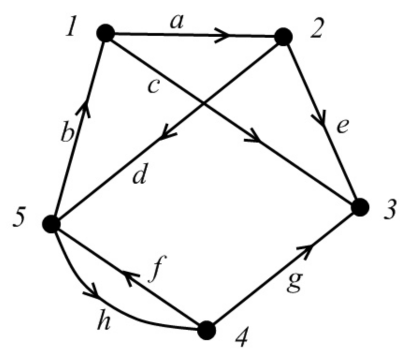

FIGURE 3

Proof. Exercise! □

Now the evaluation of $\det L(v|v)$ proceeds in two steps: first we find the combinatorial meaning of all the subdeterminants of the signed incidence matrix D, and then we feed this information into a general fact of linear algebra – the Binet-Cauchy Formula.

A little more notation is needed. Let M be a matrix, let I be a set of row-indices of M, and let J be a set of column-indices of M. Generalizing the above convention, we let $M(I|J)$ denote the submatrix of M obtained by deleting the rows in I and the columns in J from M. Also, we let $M[I|J]$ denote the submatrix of M obtained by deleting the rows **not** in I and the columns **not** in J from M. The other two possibilities, $M(I|J]$ and $M[I|J)$, are interpreted accordingly. In particular, $M(|J]$ indicates that we use all rows of M but only the columns of M in the set J.

(To follow the proof of Proposition 3 it might help to work through some examples based on the graph of Figure 3 and the corresponding signed incidence matrix.)

Proposition 3. *Let $G = (V, E)$ be a graph, orient G arbitrarily, and let D be the corresponding signed incidence matrix. Let $R \subseteq V$ and $S \subseteq E$ be such that $|R| + |S| = |V|$. Then $D(R|S]$ is square, and* $\det D(R|S] = 0$ *unless $F = (V, S)$ is a spanning forest of G in which each component contains exactly one vertex in R. Moreover, if F is such a forest then* $\det D(R|S] = \pm 1$.

Proof. Clearly $D(R|S]$ is square, with $|V| - |R|$ rows and $|S| = |V| - |R|$ columns.

First, assume that $F = (V, S)$ contains a cycle C. Orient the edges of C consistently in one of the two strongly connected orientations of

C, and for each $e \in E(C)$ let

$$\varepsilon_C(e) = \begin{cases} +1 & \text{if the orientations of } e \text{ in } C \text{ and } G \text{ agree,} \\ -1 & \text{if the orientations of } e \text{ in } C \text{ and } G \text{ differ.} \end{cases}$$

Then

$$\sum_{e \in E(C)} \varepsilon_C(e) D(|e] = \mathbf{0},$$

the zero vector. Thus, the columns of D indexed by $E(C)$ are linearly dependent. Since $E(C) \subseteq S$, the columns of D indexed by S are linearly dependent. Therefore the columns of $D(R|S]$ are linearly dependent. Therefore $\det D(R|S] = 0$.

Now assume that $F = (V, S)$ is a spanning forest of G. If some component T of F contains two distinct vertices $r_1, r_2 \in R$, then let P be the (unique) directed path from r_1 to r_2 in T. For each $e \in E(P)$ let

$$\varepsilon_P(e) = \begin{cases} +1 & \text{if the orientations of } e \text{ in } P \text{ and } G \text{ agree,} \\ -1 & \text{if the orientations of } e \text{ in } P \text{ and } G \text{ differ.} \end{cases}$$

Then

$$\sum_{e \in E(P)} \varepsilon_P(e) D(r_1 r_2|e] = \mathbf{0},$$

the zero vector. Thus, the columns of $D(r_1 r_2|)$ indexed by $E(P)$ are linearly dependent. Since $E(P) \subseteq S$ and $\{r_1, r_2\} \subseteq R$, the columns of $D(R|S]$ are linearly dependent. Therefore $\det D(R|S] = 0$. Thus, $\det D(R|S] = 0$ unless each component of F contains at most one vertex in R. But F is a forest with $|V|$ vertices and $|S|$ edges, so it has $|V| - |S| = |R|$ components. Thus, if each component of F contains at most one vertex in R then each component of F contains exactly one vertex in R.

Finally, let $F = (V, S)$ be a spanning forest of G such that each component of F contains exactly one vertex in R. Permuting the rows and columns of $D(R|S]$ as needed brings this matrix into the block diagonal form

$$M_1 \oplus M_2 \oplus \cdots \oplus M_c.$$

Here, the components of F are T_1, T_2, ..., T_c, r_i is the unique vertex of R in $V(T_i)$, and M_i is the signed incidence matrix of T_i with row r_i deleted. Since

$$\det D(R|S] = \pm \prod_{i=1}^{c} \det M_i$$

it suffices to prove the following:

CLAIM: *Let $T = (V, E)$ be a tree and let $r \in V$. Orient the edges of*

T arbitrarily, and let D be the corresponding signed incidence matrix of T. Then $\det D(r|) = \pm 1$.

This claim is proved by induction on $n = |V|$. The basis is $n = 1$, since every tree has at least one vertex. In this case $D(r|)$ is a 0-by-0 matrix, so that $\det D(r|) = 1$ (by the permutation expansion of a determinant). For the induction step $n \geq 2$, so that T has at least two vertices of degree one – let v be a vertex of degree one in T other than r. There is exactly one nonzero entry in row v of $D(r|)$, corresponding to the unique edge f of T that is incident with v, and this entry is ± 1. Evaluating $\det D(r|)$ by Laplace expansion along row v, we see that

$$\det D(r|) = \pm \det D(rv|f)$$

Now $D(rv|f)$ is the signed incidence matrix of $T \smallsetminus v$ with row r deleted. By induction, $\det D(rv|f) = \pm 1$. This completes the induction step, the proof of the claim, and the proof of the proposition. □

Let M be an r-by-m matrix, and let P be an m-by-r matrix. The product MP is then a square r-by-r matrix, so its determinant is defined. The Binet-Cauchy Formula expresses this determinant in terms of the factors M and P. Since these are not necessarily square we cannot take their determinants *per se*, so something a bit more complicated is going on.

Theorem 4 (The Binet-Cauchy Formula)**.** *Let M be an r-by-m matrix, and let P be an m-by-r matrix. Then*

$$\det MP = \sum_S \det M(|S] \cdot \det P[S|)$$

in which the sum is over all r-element subsets of the column indices of M (which are the same as the row indices of P).

(Proof of this theorem is deferred until the end of the section.)

Let $\mathcal{T}(G)$ denote the set of all spanning trees of G.

Proof of the Matrix-Tree Theorem. Let $G = (V, E)$ be a graph with $|V| = n$ vertices and Laplacian matrix L. Orient the edges of G arbitrarily and let D be the corresponding signed incidence matrix. Let $v \in V$ be any vertex. Noticing that $L(v|v) = D(v|)D^{\dagger}(|v)$ (by Lemma

2), we evaluate $\det L(v|v)$ using the Binet-Cauchy Formula:

$$\begin{aligned} \det L(v|v) &= \det D(v|)D^{\dagger}(|v) \\ &= \sum_{S\subseteq E:\ |S|=n-1} \det D(v|S]\cdot \det D^{\dagger}[S|v) \\ &= \sum_{S\subseteq E:\ |S|=n-1} |\det D(v|S]|^2 \\ &= \sum_{S\subseteq E:\ (V,S)\in\mathcal{T}(G)} 1 \\ &= \varkappa(G). \end{aligned}$$

The second equality is by the Binet-Cauchy Formula, the third equality follows since $\det M^{\dagger} = \overline{\det M}$ for any (complex) square matrix M, and the fourth equality follows from Proposition 3. The fifth equality is the definition of $\varkappa(G)$. □

The Matrix–Tree Theorem can be generalized from a conclusion about the cardinality of the set of spanning trees of $G = (V, E)$ to a conclusion about the generating polynomial for this set. To define this generating polynomial, let $\mathbf{y} = \{y_e : e \in E\}$ be pairwise commuting algebraically independent indeterminates, and for $S \subseteq E$ let $\mathbf{y}^S = \prod_{e\in S} y_e$. We may identify a spanning subgraph of G with its edge-set, and define

$$T(G;\mathbf{y}) = \sum_{T\in\mathcal{T}(G)} \mathbf{y}^T.$$

This polynomial $T(G;\mathbf{y})$ is called the *spanning-tree enumerator* of G.

Theorem 5 (The Weighted Matrix-Tree Theorem). *Let $G = (V, E)$ be a graph, orient G arbitrarily, and let D be the corresponding signed incidence matrix. Let $Y = \mathrm{diag}(y_e : e \in E)$ be the E-by-E diagonal matrix of indeterminates $\mathbf{y}$. Then, for any $v \in V$,*

$$T(G;\mathbf{y}) = \det D(v|)YD^{\dagger}(|v).$$

Proof. Notice that for any subset $S \subseteq E$ of size $|S| = n - 1$,

$$\det(DY)(v|S] = \det D(v|S]\cdot \mathbf{y}^S.$$

Having made this observation, we can copy the proof of Theorem 1 almost verbatim:

$$\begin{aligned} & \det D(v|)YD^{\dagger}(|v) \\ = \ & \det \left[(DY)(v|)D^{\dagger}(|v)\right] \\ = \ & \sum_{S\subseteq E:\ |S|=n-1} \det D(v|S] \cdot \mathbf{y}^S \cdot \det D^{\dagger}[S|v) \\ = \ & \sum_{S\subseteq E:\ |S|=n-1} |\det D(v|S]|^2 \cdot \mathbf{y}^S \\ = \ & \sum_{S\subseteq E:\ (V,S)\in\mathcal{T}(G)} \mathbf{y}^S \\ = \ & T(G;\mathbf{y}). \end{aligned}$$

□

The matrix $L(G;\mathbf{y}) = DYD^{\dagger}$ is the *weighted Laplacian matrix* of G. (As in Lemma 2, this does not depend on the choice of orientation used to define D.) This matrix will play a prominent role in the next section. The Weighted Matrix-Tree Theorem can also be written

$$T(G;\mathbf{y}) = \det L(G;\mathbf{y})(v|v).$$

Theorem 6 (The Principal Minors Weighted Matrix-Tree Theorem)**.** *Let $G = (V,E)$ be a graph, and let $L(G;\mathbf{y})$ be the weighted Laplacian matrix of G. For any subset $R \subseteq V$ of vertices,*

$$\det L(G;\mathbf{y})(R|R) = \sum_F \mathbf{y}^F,$$

in which the sum is over all spanning forests F of G for which each component of F contains exactly one vertex of R.

Proof. Exercise! □

Finally, **all** the minors of the weighted Laplacian matrix of $G = (V,E)$ can be interpreted combinatorially. There are some finicky plus or minus signs that do not appear for the principal minors. To keep track of these, let $V = \{1,2,...,n\}$ and for $R \subseteq V$ define

$$\varepsilon(R) = \prod_{r\in R}(-1)^r.$$

Let $R, Q \subseteq V$ be such that $|R| = |Q| = c$, and sort the elements of these subsets: $R = \{r_1 < r_2 < \cdots < r_c\}$ and $Q = \{q_1 < q_2 < \cdots < q_c\}$. Let

F be a spanning forest in G such that each component of F contains exactly one vertex of R and exactly one vertex of Q. Thus, there is a unique permutation $\sigma_F : \{1, 2, ..., c\} \to \{1, 2, ..., c\}$ such that r_i and $q_{\sigma_F(i)}$ belong to the same component of F, for all $1 \leq i \leq c$. Define $\varepsilon(F; R, Q)$ to be the sign of this permutation σ_F.

Theorem 7 (The All-Minors Weighted Matrix-Tree Theorem)**.** *Let $G = (V, E)$ be a graph with $V = \{1, 2, ..., n\}$, and let $L(G; \mathbf{y})$ be the weighted Laplacian matrix of G. For any subsets $R, Q \subseteq V$ of vertices with $|R| = |Q|$,*

$$\det L(G; \mathbf{y})(R|Q) = \varepsilon(R)\varepsilon(Q) \sum_F \varepsilon(F; R, Q)\mathbf{y}^F,$$

in which the sum is over all spanning forests F of G for which each component of F contains exactly one vertex of R and exactly one vertex of Q.

The proof follows the above pattern – keeping track of the signs is the only new difficulty. There is also a more general version of this theorem for directed graphs. See the paper by Chaiken [5].

We conclude this section with a proof of the Binet-Cauchy Formula.

Proof of the Binet-Cauchy Formula. We proceed by induction on r. For the basis of induction, $r = 1$, M is a row vector and P is a column vector of length m. The result follows immediately in this case from the definition of matrix product. For the induction step, assume that the result is true with $r - 1$ in place of r.

Consider the Laplace expansion of $\det MP$ along row i, for some $1 \leq i \leq n$. That is,

$$\det MP = \sum_{j=1}^{r} (-1)^{i+j} (MP)_{ij} \det(MP)(i|j)$$

We average these among all $1 \leq i \leq r$ to obtain

$$\begin{aligned} \det MP &= \frac{1}{r} \sum_{i=1}^{r} \sum_{j=1}^{r} (-1)^{i+j} (MP)_{ij} \det(MP)(i|j) \\ &= \frac{1}{r} \sum_{i=1}^{r} \sum_{j=1}^{r} (-1)^{i+j} \left(\sum_{k=1}^{m} M_{ik} P_{kj} \right) \det M(i|) P(|j) \\ &= \frac{1}{r} \sum_{i=1}^{r} \sum_{j=1}^{r} (-1)^{i+j} \left(\sum_{k=1}^{m} M_{ik} P_{kj} \right) \sum_U \det M(i|U] \cdot \det P[U|j). \end{aligned}$$

In the second of these equalities we use the fact that $(MP)(i|j) = M(i|)P(|j)$, and in the third we use the induction hypothesis. The final summation is over all $(r-1)$-element subsets U of the column indices of M.

Continuing with the calculation, we have

$$\begin{aligned} & \det MP \\ = \quad & \frac{1}{r}\sum_{U}\sum_{k=1}^{m}\left(\sum_{i=1}^{r}(-1)^{i+k}M_{ik}\det M(i|U]\right)\left(\sum_{j=1}^{r}(-1)^{k+j}P_{kj}\det P[U|j)\right) \\ = \quad & \frac{1}{r}\sum_{U}\sum_{k=1}^{m}(-1)^{r+k}\det M(|U,k]\cdot(-1)^{r+k}\det P[U,k|) \end{aligned}$$

In the last of these lines, $M(|U,k]$ is the matrix obtained from $M(|U]$ by adjoining the column $M(|k]$ on the right side. Laplace expansion along the last column shows that

$$(-1)^{r+k}\det M(|U,k] = \sum_{i=1}^{r}(-1)^{i+k}M_{ik}\det M(i|U].$$

Similarly, $P[U,k|)$ is the matrix obtained from $P[U|)$ by adjoining the row $P[k|)$ on the bottom. Laplace expansion along the last row shows that

$$(-1)^{r+k}\det P[U,k|) = \sum_{j=1}^{r}(-1)^{k+j}P_{kj}\det P[U|j).$$

Of course, if $k \in U$ then the matrix $M(|U,k]$ has two equal columns, so that $\det(M(|U,k)] = 0$. Thus, in the two outer summations we may restrict attention to pairs (U,k) such that $k \notin U$. In effect, this sums over all r-element subsets $S = U \cup \{k\}$ of the column indices of M and counts each one r times. Since the number of column–exchanges needed to obtain $M(|S]$ from $M(|U,k]$ is equal to the number of row–exchanges needed to obtain $P[S|)$ from $P[U,k|)$, we see that

$$\det M(|U,k]\cdot\det(P[U,k|) = \det M(|S]\cdot\det P[S|).$$

Continuing with the calculation, we conclude that

$$\det MP = \sum_{S}\det(M(|S])\cdot\det(P[S|))$$

which completes the induction step, and the proof. □

II. Kirchhoff's Formula.

In 1847, Kirchhoff published a short paper [10] in which he gave a formula for the effective conductance of linear resistive electrical network. (An English translation was published by O'Toole [12] in 1958.) I'll give two proofs here. The first is essentially Kirchhoff's, but the exposition benefits from a modern perspective on linear algebra. The second proof is more combinatorial, avoiding Cramer's Rule and the Matrix-Tree Theorems, although it, too, uses linear algebra (but in a quite different way).

Let $G = (V, E)$ be a finite, connected, undirected graph (which may contain loops or multiple edges), which we think of as representing an electrical network: the edges are wires and the vertices are junctions at which the wires are connected with one another. Each edge $e \in E$ is assigned an electrical resistance $r_e > 0$, a positive real number. Given two distinct vertices $a, b \in V$, we pass an electric current through the graph G by attaching the vertices a and b to the poles of an external current source. By measuring the difference in electric potential between the vertices a and b we can then determine the effective resistance of the network G between the terminals a and b, by Ohm's Law. The amazing thing is that the result of this calculation encodes a great deal of combinatorial information about the graph G.

First of all, it turns out that the formula is more naturally expressed in terms of conductance rather than resistance: conductance is merely the reciprocal of resistance. Second, resistance and conductance are conventionally real-valued quantities, but Kirchhoff's formula remains valid for quantities taken from any field. The field $\mathbb{C}(s)$ is particularly important for LRC networks to which a time-varying source of current is applied – the variable s is conjugate to the time variable by means of Laplace transorm. In this case, the analogue of resistance is referred to as *impedance*, and the analogue of conductance is referred to as *admittance*.

The linear algebra involved in the proofs below does not depend on the field of quantities used for the admittances of the edges. Accordingly, for our purposes, an *electrical network* is a pair $(G, \mathbf{y})$ in which $G = (V, E)$ is a graph as above, and $\mathbf{y} = \{y_e : e \in E\}$ is a set of algebraically independent commuting indeterminates. In effect, we choose to work over the field $\mathbb{K} = \mathbb{C}(\mathbf{y})$ of rational functions in these indeterminates. (Your intuition will not lead you astray if you think of each

y_e as a positive real number, however.) The quantity y_e is interpreted as the admittance (or conductance) of the edge e.

In order to derive Kirchhoff's Formula we need to specify the behaviour of an electrical network precisely. This is accomplished by Ohm's Law, Kirchhoff's Current Law, and Kirchhoff's Voltage Law. All three are physically intuitive and we do not dwell on their justifications.

Ohm's Law: In a wire e with ends v and w, the current j_e flowing through e from v to w is directly proportional to the difference in electric potential $\varphi(v) - \varphi(w)$ between the ends. The constant of proportionality is the admittance y_e of the wire e. That is, $j_e = y_e(\varphi(v) - \varphi(w))$.

Kirchhoff's Current Law: In an electrical network $(G, \mathbf{y})$, at every vertex v the amount of current flowing in equals the amount of current flowing out.

Kirchhoff's Voltage Law: In an electrical network $(G, \mathbf{y})$, there is a potential function $\varphi : V \to \mathbb{K}$ such that Ohm's Law is satisfied for every wire $e \in E$, and such that the currents determined by Ohm's Law also satisfy Kirchhoff's Current Law.

(Ohm's Law and Kirchhoff's Current Law specify the system of equations governing the currents and electrical potentials in the network. Thus, logically speaking, they are the axioms or postulates of the theory. Kirchhoff's Voltage Law is of a different character – it is, in effect, a lemma stating that this system of equations is consistent.)

To measure the effective admittance of an electrical network $(G, \mathbf{y})$ between the vertices $a, b \in V$ we can do the following. Connect a and b to an external source of current and force one ampere of current through the network from b to a. Ground the vertex a so that its electric potential is $\varphi(a) = 0$. The electric potential $\varphi(b)$ is then inversely proportional to the effective admittance of G, by Ohm's Law. So a calculation of this quantity

$$\mathcal{Y}_{ab}(G; \mathbf{y}) = \frac{1}{\varphi(b)}$$

is what we seek. See Figure 4 for an illustration.

Finally, to express Kirchhoff's Formula we recall the spanning-tree enumerator of G. That is, for a graph $G = (V, E)$ and indeterminates

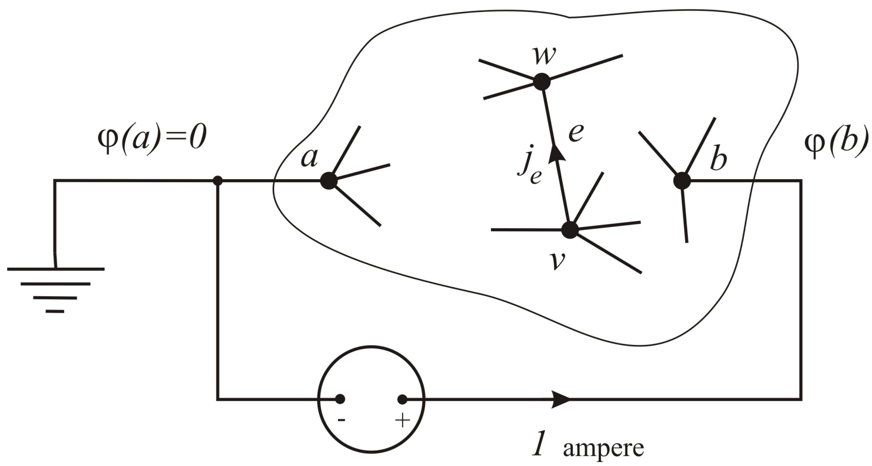

FIGURE 4. How to measure $\mathcal{Y}_{ab}(G;\mathbf{y})$.

$\mathbf{y} = \{y_e : e \in E\}$ we defined

$$T(G;\mathbf{y}) = \sum_{T \in \mathcal{T}(G)} \mathbf{y}^T,$$

in which the sum is over the set $\mathcal{T}(G)$ of all spanning trees of G. One more piece of notation is required: for a graph $G = (V, E)$ and vertices $a, b \in V$, we let G/ab denote the graph obtained by merging the two vertices a and b together into a single vertex.

Theorem 8 (Kirchhoff's Formula)**.** *Let $(G, \mathbf{y})$ be an electrical network, and let $a, b \in V$. The effective admittance between a and b in G is*

$$\mathcal{Y}_{ab}(G;\mathbf{y}) = \frac{T(G;\mathbf{y})}{T(G/ab;\mathbf{y})}.$$

First Proof of Kirchhoff's Formula. We begin by translating the physical "laws" above into linear algebra. To do this we fix an arbitrary orientation of $G = (V, E)$. (The choice of orientation does not affect the final answer, but the orientation is needed in order to write down the equations corresponding to Ohm's Law.) Next we consider the V-by-E incidence matrix D of G with respect to this orientation. Also, let $\mathbf{j} = \{j_e : e \in E\}$ be the E-indexed column vector of currents, let $Y = \mathrm{diag}(y_e : e \in E)$ be the diagonal matrix of admittances, and let $\boldsymbol{\varphi} = \{\varphi(v) : v \in V\}$ be the V-indexed column vector of voltages, normalized so that $\varphi(a) = 0$. We may restate the physical "laws" as follows:

Ohm's Law: $\mathbf{j} = -YD^\dagger \boldsymbol{\varphi}$.
This is the statement of Ohm's Law for every wire in the network simultaneously.

Kirchhoff's Current Law: $D\mathbf{j} = \boldsymbol{\delta}_a - \boldsymbol{\delta}_b$.
Here $\boldsymbol{\delta}_v$ is the V-indexed column vector given by

$$(\boldsymbol{\delta}_v)_w = \begin{cases} 1 & \text{if } w = v, \\ 0 & \text{if } w \neq v. \end{cases}$$

The reason that the RHS is not zero is that one ampere of current is being supplied to b externally and removed from a externally. The currents internal to the network G must compensate for this external driving current.

Kirchhoff's Voltage Law: A solution $\boldsymbol{\varphi}$ to Ohm's Law and Kirchhoff's Current Law exists.

Combining these equations, our task is now to solve the system

$$DYD^\dagger \boldsymbol{\varphi} = \boldsymbol{\delta}_b - \boldsymbol{\delta}_a$$

for $\boldsymbol{\varphi}$. More precisely, we only need to determine the value $\varphi(b)$. Every column of D sums to zero, as does the RHS. Therefore, this system of linear equations is redundant and we can strike out any one of them. Since we have normalized $\varphi(a) = 0$, the a-th column of $D^\dagger$ will not contribute at all to the product $D^\dagger\boldsymbol{\varphi}$. Thus, let $D_a = D(a|)$ be the matrix obtained from D by deleting row a. We seek a solution to

$$D_a Y D_a^\dagger \boldsymbol{\varphi} = \boldsymbol{\delta}_b,$$

in which $\boldsymbol{\varphi}$ and $\boldsymbol{\delta}_b$ are now column vectors indexed by $V \smallsetminus \{a\}$.

Since we only want the value of $\varphi(b)$, Cramer's Rule is the perfect technique to use. By Theorem 5,

$$\det(D_a Y D_a^\dagger) = T(G; \mathbf{y}),$$

and this is nonzero for our (indeterminate) admittances $\mathbf{y}$, so that the system is invertible. Replacing column b of $D_a Y D_a^\dagger$ by $\boldsymbol{\delta}_b$, we obtain a matrix M with $M_{bb} = 1$ being the only nonzero entry in column b. The Laplace expansion of $\det(M)$ along column b then shows that

$$\det(M) = \det(D(ab|))YD^\dagger(|ab)) = \det L(ab|ab),$$

in which $L = L(G; \mathbf{y}) = DYD^\dagger$ is the weighted Laplacian matrix of G. By Theorem 6 this determinant is the generating series for spanning forests of G which have exactly two components, one containing a and one containing b. These forests of G correspond bijectively with spanning trees of G/ab, so that $\det(M) = T(G/ab; \mathbf{y})$. Cramer's Rule thus implies that

$$\varphi(b) = \frac{T(G/ab; \mathbf{y})}{T(G; \mathbf{y})}.$$

Since the effective admittance was defined to be $\mathcal{Y}_{ab}(G; \mathbf{y}) = 1/\varphi(b)$, this completes the proof. □

Second Proof of Kirchhoff's Formula. Recall that $\mathbb{K} = \mathbb{C}(\mathbf{y})$ is the field of rational functions in the indeterminates $\mathbf{y} = \{y_e : e \in E\}$ and that $\mathcal{T}(G)$ is the set of spanning trees of G. Let $\mathbb{K}\mathcal{T}(G)$ denote the vector space over $\mathbb{K}$ which has as a basis $\{[T] : T \in \mathcal{T}(G)\}$. That is, a vector in $\mathbb{K}\mathcal{T}(G)$ is a formal linear combination

$$\sum_{T \in \mathcal{T}(G)} c_T \cdot [T]$$

in which the coefficients c_T are in $\mathbb{K}$. The vector space $\mathbb{K}\mathcal{T}(G/ab)$ is defined similarly. We are going to define a linear transformation $Q : \mathbb{K}\mathcal{T}(G) \to \mathbb{K}\mathcal{T}(G/ab)$, the properties of which will allow us to prove Kirchhoff's Formula. In order to define Q we fix an arbitrary orientation for each edge $e \in E$, just as in the first proof. For each $e \in E$, let head(e) denote the vertex into which e points, and let tail(e) denote the vertex out of which e points.

It suffices to define the action of Q on each basis vector $[T]$ for $T \in \mathcal{T}(G)$. In the spanning tree T of G, there is a unique directed path $P(T)$ which begins at b and ends at a. For each edge $e \in P(T)$, say that the sign of (T, e) is

$$\varepsilon(T, e) = \begin{cases} +1 & \text{if the orientations of } e \text{ in } P(T) \text{ and } G \text{ agree,} \\ -1 & \text{if the orientations of } e \text{ in } P(T) \text{ and } G \text{ differ.} \end{cases}$$

We define $Q([T])$ by the formula

$$Q([T]) = \sum_{e \in P(T)} \varepsilon(T, e) j_e [T \smallsetminus e],$$

in which $\mathbf{j} = \{j_e : e \in E\}$ is the vector of currents as in the first proof. Notice that as a set of edges, $T \smallsetminus e$ is a spanning tree of G/ab for every $e \in P(T)$. This action of Q is extended linearly to all vectors in $\mathbb{K}\mathcal{T}(G)$.

Now consider the vector $\mathbf{g} = \sum_{T \in \mathcal{T}(G)} [T]$ in $\mathbb{K}\mathcal{T}(G)$ – that is, the formal sum of all spanning trees of G. Then $Q(\mathbf{g})$ is some vector in

$\mathbb{K}\mathfrak{T}(G/ab)$, so that

$$Q(\mathbf{g}) = \sum_{Z \in \mathfrak{T}(G/ab)} c_Z[Z]$$

for some coefficients $c_Z \in \mathbb{K}$. By the definition of Q, we see that

$$c_Z = \sum_{T \in \mathfrak{T}(G):\ T \smallsetminus e = Z} \varepsilon(T, e) j_e.$$

Regarding Z as a forest in G with two components A containing a and B containing b, a spanning tree $T \in \mathfrak{T}(G)$ is such that $T \smallsetminus e = Z$ if and only if $T = Z \cup \{e\}$ and e is an edge of G which has one end in A and the other end in B. Let $C(Z)$ be this set of edges of G, so that

$$c_Z = \sum_{e \in C(Z)} \varepsilon(Z \cup \{e\}, e) j_e.$$

Notice that the signs $\varepsilon(Z \cup \{e\}, e)$ are such that this sum is the total current flowing from B to A through the wires in the set $C(Z)$. Applying Kirchhoff's Current Law to every vertex of B, and remembering that one ampere of current is entering $b \in B$ from an external source, we see that the total current flowing from B to A through $C(Z)$ is also one ampere. That is, $c_Z = 1$ for every $Z \in \mathfrak{T}(G/ab)$, so that

$$Q(\mathbf{g}) = \sum_{Z \in \mathfrak{T}(G/ab)} [Z].$$

This is an equation between vectors in the vector space $\mathbb{K}\mathfrak{T}(G/ab)$.

Next, we define a linear functional $\alpha : \mathbb{K}\mathfrak{T}(G/ab) \to \mathbb{K}$ as follows: for $Z \in \mathfrak{T}(G/ab)$, let

$$\alpha([Z]) = \mathbf{y}^Z$$

and extend this linearly to all of $\mathbb{K}\mathfrak{T}(G/ab)$. We examine the result of applying this linear functional to both sides of the vector equation derived in the previous paragraph. On the RHS we obtain

$$\alpha\left(\sum_{Z \in \mathfrak{T}(G/ab)} [Z]\right) = \sum_{Z \in \mathfrak{T}(G/ab)} \mathbf{y}^Z = T(G/ab; \mathbf{y}),$$

which is the spanning-tree enumerator of G/ab. For the LHS, let's first consider some $T \in \mathfrak{T}(G)$ and compute $\alpha(Q([T]))$. That is, using Ohm's

Law,

$$\begin{aligned}
\alpha(Q([T])) &= \sum_{e\in P(T)} \varepsilon(T,e) j_e \mathbf{y}^{T\smallsetminus e} \\
&= \sum_{e\in P(T)} \varepsilon(T,e) y_e \left(\varphi(\mathrm{tail}(e)) - \varphi(\mathrm{head}(e))\right) \mathbf{y}^{T\smallsetminus e} \\
&= \mathbf{y}^T \sum_{e\in P(T)} \varepsilon(T,e) \left(\varphi(\mathrm{tail}(e)) - \varphi(\mathrm{head}(e))\right) \\
&= \mathbf{y}^T \left(\varphi(b) - \varphi(a)\right) = \varphi(b)\mathbf{y}^T,
\end{aligned}$$

since $\varphi(a) = 0$. The penultimate equality follows because the signs $\varepsilon(T,e)$ are such that each vertex v on $P(T)$ other than a or b will contribute both $+\varphi(v)$ and $-\varphi(v)$ to the summation, so the summation of differences "telescopes". Consequently, we have

$$\alpha(Q(\mathbf{g})) = \sum_{T\in\mathcal{T}(G)} \varphi(b)\mathbf{y}^T = \varphi(b)T(G;\mathbf{y}).$$

Comparing this with the RHS we obtain the equation of rational functions

$$\varphi(b)T(G;\mathbf{y}) = T(G/ab;\mathbf{y}).$$

Finally, we conclude that

$$\mathcal{Y}_{ab}(G;\mathbf{y}) = \frac{1}{\varphi(b)} = \frac{T(G;\mathbf{y})}{T(G/ab;\mathbf{y})},$$

which completes the proof. □

It is not hard to adapt the first proof of Kirchhoff's Formula to compute the electrical potentials $\varphi(v)$ for all $v \in V$. (In doing so, the All-Minors Matrix-Tree Theorem comes into play.) From these electrical potentials, all of the currents are then determined by Ohm's Law, and they turn out to have simple expressions as determinants as well. This determines the response of the whole network to a simple signal consisting of a current flowing in at a single vertex and out at another single vertex. The response of the network to a more complicated applied current can then be calculated by the "Principle of Superposition", since the system of equations is linear. It is also possible to solve the system while holding some of the vertices at fixed electrical potentials (subject to certain restrictions).

For a thorough development of the theory of electrical networks I recommend the reference text of Balabanian and Bickart [1]. Vágó's book [16] is a good introduction. The book by Doyle and Snell [8] is an excellent, relatively non-technical, development of some of the phenomena discussed in Sections III and IV below. Biggs' paper [2] and Chapter 2 of Lyons and Peres [11] develop this theory in more detail. The classic paper of Brooks, Smith, Stone, and Tutte [4] gives a surprising application of electrical network theory in combinatorics.

III. The Half-Plane Property and Rayleigh Monotonicity.

In this section we prove two physically intuitive properties of electrical networks: Rayleigh Monotonicity and the Half-Plane Property. Somewhat more precisely, we show that every electrical network satisfies the half-plane property, and that every homogeneous, multiaffine polynomial satisfying the half-plane property also satisfies Rayleigh monotonicity. Then we abstract these properties and apply them to structures more general than graphs. As a consequence we deduce some inequalities for spanning trees in graphs (and for bases in certain more general matroids).

Let $G = (V, E)$ be a connected graph, let $\mathbf{y} = \{y_e : \ e \in E\}$ be indeterminate admittances on the edges of G, and let $a, b \in V$ be distinct vertices of G. Let $\mathcal{Y}_{ab}(G; \mathbf{y})$ denote the effective admittance of the network $(G, \mathbf{y})$ measured between a and b. By Kirchhoff's Formula

$$\mathcal{Y}_{ab}(G; \mathbf{y}) = \frac{T(G; \mathbf{y})}{T(G/ab; \mathbf{y})}.$$

Let H be the graph obtained from G by adjoining a new edge f with ends a and b. Then $G = H \smallsetminus f$ and $G/ab = H/f$, so that Kirchhoff's Formula can also be written

$$\mathcal{Y}_{ab}(G; \mathbf{y}) = \frac{T(H \smallsetminus f; \mathbf{y})}{T(H/f; \mathbf{y})}.$$

We begin by considering Rayleigh Monotonicity. Let $e \in E$ be any edge of G. If each edge of G has positive admittance ($y_c > 0$ for all $c \in E$) and if the admittance y_e is increased, then the effective admittance $\mathcal{Y}_{ab}(G; \mathbf{y})$ cannot decrease: this is the principle of Rayleigh Monotonicity. (It is physically sensible but does require proof.) Expressed symbolically, if $y_c > 0$ for all $c \in E$ then

$$\frac{\partial}{\partial y_e} \mathcal{Y}_{ab}(G; \mathbf{y}) \geq 0.$$

Now for some shorthand notation: let $T = T(H; \mathbf{y})$ and for $c \in E(H)$ let $T^c = T(H \smallsetminus c; \mathbf{y})$ and $T_c = T(H/c; \mathbf{y})$. Thus we have

$$T = T^c + y_c T_c$$

for any $c \in E(H)$. This is a generalization of the deletion/contraction recursion at the beginning of Section I. We extend this notation to multiple (but distinct) sub- and super-scripts in the obvious way. In particular,

$$T = T^{ef} + y_e T_e^f + y_f T_f^e + y_e y_f T_{ef}.$$

(Note that the polynomials T^{ef}, T_e^f, T_f^e and T_{ef} do not involve the indeterminates y_e or y_f.)

Computing the above partial derivative by Kirchhoff's Formula and the quotient rule yields

$$\begin{aligned} \frac{\partial}{\partial y_e}\mathcal{Y}_{ab}(G;\mathbf{y}) &= \frac{\partial}{\partial y_e}\frac{T^f}{T_f} = \frac{\partial}{\partial y_e}\frac{T^{ef} + y_e T_e^f}{T_f^e + y_e T_{ef}} \\ &= \frac{T_e^f(T_f^e + y_e T_{ef}) - (T^{ef} + y_e T_e^f)T_{ef}}{(T_f^e + y_e T_{ef})^2} \\ &= \frac{T_e^f T_f^e - T_{ef}T^{ef}}{(T_f)^2}. \end{aligned}$$

Thus, the principle of Rayleigh Monotonicity is equivalent to the assertion that for any graph H with positive edge-weights $\{y_c : c \in E(H)\}$, and for any two edges $e, f \in E(H)$,

$$\Delta T(H)\{e,f\} = T_e^f T_f^e - T_{ef}T^{ef} \geq 0,$$

in which $T = T(H) = T(H;\mathbf{y})$ is the spanning-tree enumerator of H. This difference $\Delta T(H)\{e,f\}$ is called the *Rayleigh difference of e and f in $T(H)$*. We will derive Rayleigh monotonicity in this form from the half-plane property.

The motivation for the Half-Plane Property is also physical, but is less intuitive. To really understand it, one needs to know how the Laplace transform is used to determine the effective admittance of a linear LRC network – I will only describe this informally. In a network with inductors and capacitors as well as resistors one considers all the currents and electrical potentials as functions of time. Ohm's Law must be generalized to describe the inductors and capacitors – the result is a linear system of coupled integro-differential equations for these quantities. Miraculously, the Laplace transform maps this into a system of linear equations for rational functions in the field $\mathbb{C}(s)$, which can be solved as in Section II. The variable $s = \lambda + i\omega$ is Laplace conjugate to the time variable, and physically is a *complex frequency*. The imaginary part ω is a phase-shift, and the real part governs the amplitude response of the network element: $\lambda > 0$ corresponds to decreasing amplitude, and $\lambda < 0$ corresponds to increasing amplitude.

A *passive* network element is one which dissipates energy, or in other words does not contain an internal power source. Ordinary resistors, capacitors, and inductors are passive elements. Since a passive network element $e \in E$ dissipates energy, its amplitude response is decreasing, so that the real part of its admittance is positive: $\mathrm{Re}(y_e) > 0$. It is physically sensible that if every element $e \in E$ dissipates energy, then the whole network $(G, \mathbf{y})$ dissipates energy. That is, if $\mathrm{Re}(y_e) > 0$ for all $e \in E$, then $\mathrm{Re}\, \mathcal{Y}_{ab}(G; \mathbf{y}) > 0$. This is the Half-Plane Property.

As with Rayleigh monotonicity, we translate the half-plane property into an equivalent statement about the spanning-tree enumerator $T(H; \mathbf{y})$. Assume that $\mathbf{y} = \{y_c : \ c \in E(H)\}$ are complex numbers such that $T(H; \mathbf{y}) = 0$. Since $T = T^f + y_f T_f = 0$, we see that

$$y_f = -\frac{T^f}{T_f} = -\mathcal{Y}_{ab}(G; \mathbf{y}).$$

If $\mathrm{Re}(y_c) > 0$ for all $c \in E(G)$, then the half-plane property implies that $\mathrm{Re}\, \mathcal{Y}_{ab}(G; \mathbf{y}) > 0$, so that $\mathrm{Re}(y_f) < 0$. It follows from this that if $\mathrm{Re}(y_c) > 0$ for all $c \in E(H)$, then $T(H; \mathbf{y}) \neq 0$. It is this property which we call the *half-plane property* of the polynomial $T(H; \mathbf{y})$.

Conversely, from the half-plane property of the spanning tree enumerator, it follows that if $\mathrm{Re}(y_c) > 0$ for all $c \in E(G)$, then $\mathrm{Re}\, \mathcal{Y}_{ab}(G; \mathbf{y}) \geq 0$. To show that the real part of the effective admittance is in fact **strictly** positive depends on the fact that $T(G; \mathbf{y})$ is homogeneous – to prove this we need to develop a little bit of the general theory first.

To summarize, we have identified two properties – Rayleigh Monotonicity and the Half-Plane Property – that hold for all electrical networks. In fact, we can abstract these properties away from this particular physical application, and consider whether they hold more generally. To do this, let $Z \in \mathbb{C}[\mathbf{y}]$ be any polynomial in the indeterminates $\mathbf{y} = \{y_e : \ e \in E\}$ with complex coefficients. Such a polynomial is *homogeneous* if every monomial occurring in Z has the same total degree, and is *multiaffine* if every indeterminate y_e occurs at most to the first power in Z. For example, the spanning-tree enumerator of a graph is homogeneous and multiaffine.

We can use the sub- and super-script notation for any multiaffine polynomial Z. For example, for any $e, f \in E$,

$$Z = Z^{ef} + y_e Z_e^f + y_f Z_f^e + y_e y_f Z_{ef}.$$

We say that Z satisfies the *Rayleigh condition* provided that all coefficients of Z are nonnegative, and if $y_c > 0$ for all $c \in E$, then for any distinct $e, f \in E$,

$$\Delta Z\{e, f\} = Z_e^f Z_f^e - Z_{ef} Z^{ef} \geq 0.$$

We say that Z has the *half-plane property* provided that either Z is identically zero, or if $\mathrm{Re}(y_c) > 0$ for all $c \in E$, then $Z(\mathbf{y}) \neq 0$. It will be convenient to have the notation

$$\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$$

for the open right half-plane in $\mathbb{C}$.

Theorem 9 is a "folklore" result in Electrical Engineering.

Theorem 9. *For any graph $G = (V, E)$, the spanning-tree enumerator $T(G; \mathbf{y})$ has the half-plane property.*

Proof. Orient G arbitrarily, and let D be the corresponding signed incidence matrix. Let $a \in V$ be any vertex, and let $D_a = D(a|)$. By Proposition 3, the rows of D_a are linearly independent. By Theorem 5, $T(G; \mathbf{y}) = \det L(a|a) = \det D_a Y D_a^\dagger$.

Now let $y_e \in \mathbb{C}$ be such that $\mathrm{Re}(y_e) > 0$ for all $e \in E$. We will show that $T(G; \mathbf{y}) \neq 0$ by showing that $L(a|a)$ is an invertible matrix, so that $\det L(a|a) \neq 0$. To see that $L(a|a)$ is invertible, consider any nonzero column vector $\mathbf{z} \neq \mathbf{0}$ indexed by $V \smallsetminus \{a\}$. We will show that $L(a|a)\mathbf{z} \neq \mathbf{0}$: this is one criterion for $L(a|a)$ to be invertible. To show that $L(a|a)\mathbf{z} \neq \mathbf{0}$, we will show that $\mathbf{z}^\dagger L(a|a)\mathbf{z} \neq 0$. Let $\mathbf{w} = D_a^\dagger \mathbf{z}$. The columns of $D_a^\dagger$ are linearly independent, and $\mathbf{z} \neq \mathbf{0}$, so that $\mathbf{w} \neq \mathbf{0}$. Now

$$\mathbf{z}^\dagger L(a|a)\mathbf{z} = \mathbf{z}^\dagger D_a Y D_a^\dagger \mathbf{z} = \mathbf{w}^\dagger Y \mathbf{w} = \sum_{e \in E} y_e |w_e|^2.$$

The numbers $|w_e|^2$ are nonnegative real numbers, and at least one of them is strictly positive since $\mathbf{w} \neq \mathbf{0}$. Since $\mathrm{Re}(y_e) > 0$ for all $e \in E$ it follows that $Re(\mathbf{z}^\dagger L(a|a)\mathbf{z}) > 0$: in particular, $\mathbf{z}^\dagger L(a|a)\mathbf{z} \neq 0$. Therefore, $L(a|a)$ is invertible, so that $\det L(a|a) \neq 0$. This completes the proof. □

In order to derive Rayleigh monotonicity from the half-plane property we make use of the following general facts. The proofs are a bit finicky, but not really difficult.

Lemma 10. *Let Z be a multiaffine polynomial with the half-plane property, let $e \in E$, and consider $Z = Z^e + y_e Z_e$. Then both Z^e and Z_e also have the half-plane property. Thus, for disjoint subsets $I, J \subseteq E$, the polynomial Z_I^J has the half-plane property.*

Proof. If $Z \equiv 0$ then the statement is trivial, so assume not. We begin by showing that Z_e has the half-plane property. If $Z_e \equiv 0$ then this is trivial, so assume not. Let $w_c \in \mathbb{C}$ be such that $\mathrm{Re}(w_c) > 0$ for all

$c \in E \smallsetminus e$. Since Z has the half-plane property, for any such $\mathbf{w}$ we have $\mathrm{Re}(Z^e(\mathbf{w})/Z_e(\mathbf{w})) \geq 0$ if $Z_e(\mathbf{w}) \neq 0$, and if $Z_e(\mathbf{w}) = 0$ then $Z^e(\mathbf{w}) \neq 0$.

Suppose that $\mathbf{w} \in \mathcal{H}^{E \smallsetminus e}$ is such that $Z_e(\mathbf{w}) = 0$. Since $Z_e \not\equiv 0$, there is an index $f \in E \smallsetminus e$ such that $Z_{ef}(\mathbf{w}) \neq 0$ (and hence $Z^e_f(\mathbf{w}) \neq 0$). Now let $y_c = w_c$ for all $c \in E \smallsetminus \{e, f\}$ and let $y_f = w_f + \eta$, where $\eta = \rho e^{i\theta}$ with $\rho > 0$ small enough so that $\mathrm{Re}(y_f) > 0$ for all $0 \leq \theta < 2\pi$. (That is, $\rho < \mathrm{Re}(w_f)$.) As a polynomial in η, $Z_e(\mathbf{y}) = 0$ when $\eta = 0$. Since this polynomial has only finitely many roots we can choose a small ρ such that $Z_e(\mathbf{y}) \neq 0$ for all $0 \leq \theta < 2\pi$. Now

$$\frac{Z^e(\mathbf{y})}{Z_e(\mathbf{y})} = \frac{Z^e(\mathbf{w}) + \eta Z^e_f(\mathbf{w})}{Z_e(\mathbf{w}) + \eta Z_{ef}(\mathbf{w})} = \frac{Z^e(\mathbf{w})}{Z_{ef}(\mathbf{w})} \cdot \frac{1}{\eta} + \frac{Z^e_f(\mathbf{w})}{Z_{ef}(\mathbf{w})}.$$

Note that since $Z_e(\mathbf{w}) = 0$ we have $Z^e(\mathbf{w}) \neq 0$, so the first term is not zero. Take $\rho > 0$ small enough that the modulus of the first term is strictly greater than the modulus of the second term (and $\mathrm{Re}(y_f) > 0$, too, of course). Since $0 \leq \theta < 2\pi$ is arbitrary, we can choose θ so that $\mathrm{Re}(Z^e(\mathbf{y})/Z_e(\mathbf{y})) < 0$. But this contradicts the fact that Z has the half-plane property. Therefore, the point $\mathbf{w}$ does not exist, so that Z_e has the half-plane property.

Replacing y_e by $1/y_e$ in Z and multiplying by y_e, one sees that since $Z = Z^e + y_e Z_e$ has the half-plane property, so does the polynomial $Z_e + y_e Z^e$. (This polynomial need not be homogeneous.) Applying the above argument to this polynomial shows that Z^e has the half-plane property. The remaining statement about Z^J_I follows by induction on $|I| + |J|$. □

(With Lemma 10 at hand, we can now show that the half-plane property for $T = T(H; \mathbf{y})$ implies that $\mathrm{Re}\, \mathcal{Y}_{ab}(G; \mathbf{y}) > 0$ for all $\mathbf{y} \in \mathcal{H}^{E(G)}$. We have seen that $\mathrm{Re}\, T^f(\mathbf{y})/T_f(\mathbf{y}) \geq 0$ for such $\mathbf{y}$. From the form of T we see that neither T^f nor T_f is identically zero. By Lemma 10, both T^f and T_f have the half-plane property, so that T^f/T_f defines a (nonvanishing) holomorphic function from $\mathcal{H}^{E(G)}$ to $\mathbb{C}$. By the Open Mapping Theorem, the image of this function is either a single point or an open subset of $\mathbb{C}$. Since T is homogeneous, the polynomials T^f and T_f have different degrees, so that T^f/T_f is not a constant function. Its image is therefore open and contained in the closure of $\mathcal{H}$. Thus, its image is contained in $\mathcal{H}$, so that $\mathrm{Re}\, \mathcal{Y}_{ab}(G; \mathbf{y}) > 0$ for all $\mathbf{y} \in \mathcal{H}^{E(G)}$.)

Proposition 11. *Let $Z \in \mathbb{C}[\mathbf{y}]$ be a homogeneous multiaffine polynomial. The following are equivalent:*
(a) *Z has the half-plane property.*

(b) *If* $a_e \geq 0$ *and* $b_e \geq 0$ *and* $y_e = a_e x + b_e$ *for all* $e \in E$*, then as a polynomial in* x*,* $Z(\mathbf{a}x + \mathbf{b})$ *has only real nonpositive roots.*

Proof. Let Z be homogeneous of degree r. To see that (a) implies (b), assume that Z has the half-plane property, and let $y_e = a_e x + b_e$ as in part (b). Let J be the set of indices $e \in E$ for which $y_e = 0$, and note that $Z(\mathbf{y}) = Z^J(\mathbf{y})$. Let $\xi \in \mathbb{C}$ be any complex number that is not a nonpositive real. Then there are complex numbers $z, w \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$, $\mathrm{Re}(w) > 0$, and $z/w = \xi$. Let $u_e = a_e z + b_e w$, and note that $u_e = 0$ for $e \in J$, and $\mathrm{Re}(u_e) > 0$ for $e \in E \smallsetminus J$. Since Z^J has the half-plane property (by Lemma 10) we see that $Z(\mathbf{u}) = Z^J(\mathbf{u}) \neq 0$. Finally,

$$Z(\mathbf{a}\xi + \mathbf{b}) = w^{-r}\, Z(\mathbf{u}) \neq 0,$$

so that the polynomial $Z(\mathbf{a}x + \mathbf{b})$ has only real nonpositive roots. This shows that (a) implies (b).

Conversely, assume (b) and let $y_e \in \mathbb{C}$ be such that $\mathrm{Re}(y_e) > 0$ for all $e \in E$. Then there are complex numbers $z, w \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$ such that all of the y_e are nonnegative linear combinations of z and w: that is, $y_e = a_e z + b_e w$ with $a_e \geq 0$ and $b_e \geq 0$ for all $e \in E$. (For instance, let z and w be the elements of the set $\{y_e : \ e \in E\}$ of maximum or of minimum argument, respectively.) Since $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$, it follows that $\xi = z/w$ is not a nonpositive real number. Therefore, from (b) it follows that $Z(\mathbf{a}\xi + \mathbf{b}) \neq 0$, and hence that

$$Z(\mathbf{y}) = Z(\mathbf{a}z + \mathbf{b}w) = w^r\, Z(\mathbf{a}\xi + \mathbf{b}) \neq 0,$$

so that Z has the half-plane property, proving (a). □

Lemma 12. *Let* $Z \in \mathbb{C}[\mathbf{y}]$ *be a homogeneous multiaffine polynomial with the half-plane property. For any distinct* $e, f \in E$*, if neither of the polynomials* Z^{ef} *or* Z_{ef} *is identically zero, then neither of the polynomials* Z_e^f *or* Z_f^e *is identically zero.*

Proof. Suppose that neither of Z^{ef} or Z_{ef} is identically zero, but that at least one of Z_e^f or Z_f^e is identically zero. By symmetry, we may assume that $Z_e^f \equiv 0$. We will derive a contradiction by finding $y_c \in \mathbb{C}$ with $\mathrm{Re}(y_c) > 0$ for all $c \in E$ such that $Z(\mathbf{y}) = 0$. First, we solve the equation

$$0 = Z(\mathbf{y}) = Z^{ef} + y_f Z_f^e + y_e y_f Z_{ef}$$

for the variable y_e to get

$$y_e = -\frac{Z^{ef}}{Z_{ef}} \cdot \frac{1}{y_f} - \frac{Z_f^e}{Z_{ef}}.$$

Now Z^{ef}/Z_{ef} is a nonconstant rational function which, by Lemma 10, is nonvanishing and holomorphic on the connected open set $\mathcal{H}^{E\smallsetminus\{e,f\}}$. By the Open Mapping Theorem, the image of this set by this function is an open subset of $\mathbb{C}$, so it is not contained in the nonnegative real axis. Thus, there are complex numbers $y_c \in \mathbb{C}$ with $\mathrm{Re}(y_c) > 0$ for all $c \in E \smallsetminus \{e, f\}$ such that Z^{ef}/Z_{ef} is not a nonnegative real number. Now we can choose $y_f = \rho e^{i\theta}$ with $\rho > 0$ and $-\pi/2 < \theta < \pi/2$ so that $-Z^{ef}/Z_{ef}y_f$ has strictly positive real part. Finally, taking $\rho > 0$ sufficiently small we can guarantee that $\mathrm{Re}(-Z^{ef}/Z_{ef}y_f) > \mathrm{Re}(-Z_f^e/Z_{ef})$. This implies that $\mathrm{Re}(y_e) > 0$, and gives a solution to $Z(\mathbf{y}) = 0$ which contradicts the fact that Z has the half-plane property. This shows that neither Z_e^f nor Z_f^e are identically zero. □

Theorem 13. *Let $Z \in \mathbb{R}[\mathbf{y}]$ be a homogeneous multiaffine polynomial with nonnegative coefficients. If Z has the half-plane property then Z satisfies Rayleigh monotonicity.*

Proof. Assume that Z has the half-plane property, and let $y_c > 0$ for all $c \in E$. Consider two distinct indices $e, f \in E$, and the expansion

$$Z(\mathbf{y}) = Z^{ef}(\mathbf{y}) + y_e Z_e^f(\mathbf{y}) + y_f Z_f^e(\mathbf{y}) + y_e y_f Z_{ef}(\mathbf{y}).$$

The coefficients Z^{ef}, Z_e^f, Z_f^e, and Z_{ef} are nonnegative real numbers. Since Z has nonnegative coefficients and $y_e > 0$ for all $e \in E$, these coefficients are zero only if they are identically zero as polynomials. By Lemma 12, if either $Z_e^f = 0$ or $Z_f^e = 0$ then either $Z^{ef} = 0$ or $Z_{ef} = 0$.

Now let $w_c = a_c x + b_c$ for each $c \in E$, in which

$$a_c = \begin{cases} Z_f^e & \text{if } c = e, \\ Z_e^f & \text{if } c = f, \\ 0 & \text{if } c \in E \smallsetminus \{e, f\} \end{cases}$$

and

$$b_c = \begin{cases} 0 & \text{if } c = e, \\ 0 & \text{if } c = f, \\ y_c & \text{if } c \in E \smallsetminus \{e, f\} \end{cases}$$

By Proposition 11, $Z(\mathbf{w}) = Z(\mathbf{a}x + \mathbf{b})$ has only real nonpositive roots. Substituting into the above expansion, we see that

$$Z(\mathbf{w}) = Z^{ef} + (2Z_e^f Z_f^e)x + (Z_e^f Z_f^e Z_{ef})x^2.$$

Since this quadratic in x has only real roots, its discriminant is nonnegative. That is,

$$(2Z_e^f Z_f^e)^2 - 4(Z_e^f Z_f^e Z_{ef})Z^{ef} \geq 0.$$

Finally, if either $Z_e^f = 0$ or $Z_f^e = 0$ then, since either $Z^{ef} = 0$ or $Z_{ef} = 0$, we see that $\Delta Z\{e,f\} = 0$. If both $Z_e^f > 0$ and $Z_f^e > 0$ then the nonnegativity of the above discriminant yields $\Delta Z\{e,f\} \geq 0$. This shows that Z satisfies Rayleigh monotonicity. □

This is not the simplest proof of Rayleigh monotonicity for electrical networks. A very simple proof *via* "Thomson's Principle" is given in Sections 3.5 and 4.1 of Doyle and Snell [8], and in Chapter 2 of Lyons and Peres [11]. The fact that the half-plane property implies Rayleigh monotonicity was sharpened by Brändén [3], who showed that a homogeneous multiaffine polynomial Z with nonnegative coefficients has the half-plane property **if and only if** it is *strongly Rayleigh*, meaning that for any two $e, f \in E$, $\Delta Z\{e,f\} \geq 0$ whenever all $y_c \in \mathbb{R}$ are real (not just positive).

* * * * *

We now turn to some combinatorial applications of this theory of electrical networks. This is motivated by the following two questions. What do we need to assume to get a (homogeneous, multiaffine) polynomial $Z \in \mathbb{C}[\mathbf{y}]$ that has the half-plane property, or satisfies Rayleigh monotonicity? What consequences can we derive from these properties? It turns out that these questions fit naturally within a branch of combinatorics called *Matroid Theory*.

Matroids were defined in 1932 by Whitney, as an abstraction of the idea of linear independence in vector spaces. The book by Oxley [13] has become the standard reference, and he has also written an excellent introductory survey [14]. There are several equivalent definitions – the most convenient for our purposes is as follows. A *matroid* is a pair $\mathcal{M} = (\mathcal{B}, E)$ in which E is a finite set and $\mathcal{B}$ is a collection of subsets $B \subseteq E$, all of the same size $|B| = r$, satisfying the following *basis exchange axiom*:

- if $B, B' \in \mathcal{B}$ and $e \in B \smallsetminus B'$, then there exists $f \in B' \smallsetminus B$ such that $(B \smallsetminus e) \cup f \in \mathcal{B}$.

A set in the collection $\mathcal{B}$ is called a *basis* of $\mathcal{M} = (\mathcal{B}, E)$, and r is the *rank* of $\mathcal{M}$. The basis exchange axiom is a familiar property of any two bases B, B' of a finite-dimensional vector space over any field.

As an example, let $\mathbb{F}$ be any field, and let A be an r-by-m matrix of rank r that has entries in $\mathbb{F}$. Let E be the set of column indices of A. This defines a matroid $\mathcal{M} = \mathcal{M}[A]$ by saying that $B \subseteq E$ is in $\mathcal{B}$ if and only if $A(|B]$ is invertible. The matrix A is a *representation of* $\mathcal{M}$ *over the field* $\mathbb{F}$. If a matroid $\mathcal{M}$ has a representation over $\mathbb{F}$ then $\mathcal{M}$ is $\mathbb{F}$*-representable* or *representable over* $\mathbb{F}$. These are the examples of matroids that come from linear algebra. There are, however, matroids that are not representable over any field.

To connect with graph theory, let $G = (V, E)$ be a connected graph, orient the edges of G arbitrarily, and let $D = D(G)$ be the corresponding signed incidence matrix. Let $a \in V$ be any vertex, and let $D_a = D(a|)$. Then D_a is an $(n-1)$-by-m matrix over $\mathbb{R}$, and for $B \subseteq E$ of size $|B| = n-1$, Proposition 3 shows that $D_a(|B]$ is invertible if and only if (V, B) is a spanning tree of G. Thus, the matroid $\mathcal{M}[D_a]$ does not depend on the choices of vertex $a \in V$ nor of orientation of G. This is the *graphic matroid* $\mathcal{M}(G)$ of G: the bases of $\mathcal{M}(G)$ are the (edge-sets of) spanning trees of G.

Proposition 3 shows that graphic matroids have a special property: they can be represented by a matrix A over $\mathbb{R}$ such that every square submatrix of A has determinant ± 1 or 0. Such matroids $\mathcal{M} = \mathcal{M}[A]$ are called *regular*. There are regular matroids that are not graphic. A famous theorem of Tutte is that a matroid is regular if and only if it can be represented over both GF(2) and GF(3). Another famous theorem of Seymour gives a structural characterization of regular matroids.

A condition weaker than being regular is as follows. A matroid $\mathcal{M}$ is *complex totally unimodular* (CTU) if and only if it can be represented by a matrix A over $\mathbb{C}$ such that every square submatrix of A has determinant either 0 or of modulus 1. A related condition is that $\mathcal{M}$ is *sixth-root-of-unity* ($\sqrt[6]{1}$), meaning that it can be represented by a matrix A over $\mathbb{C}$ such that every square submatrix of A has determinant either 0 or a sixth-root of one. Clearly, if $\mathcal{M}$ is $\sqrt[6]{1}$ then $\mathcal{M}$ is CTU. A recent theorem of Choe, Oxley, Sokal, and Wagner [6] is that the converse holds: a matroid is CTU if **and only if** it is $\sqrt[6]{1}$. Every regular matroid is $\sqrt[6]{1}$, but there are $\sqrt[6]{1}$ matroids that are not regular. Another recent theorem due to Whittle [20] is that a matroid is $\sqrt[6]{1}$ if and only if it is representable over both GF(3) and GF(4).

We can generalize the spanning-tree enumerator of a graph $G = (V, E)$ by thinking of it in terms of the graphic matroid $\mathcal{M} = \mathcal{M}(G)$. Let $\mathcal{M} = (\mathcal{B}, E)$ be any matroid, and let $\mathbf{y} = \{y_e : e \in E\}$ be algebraically independent commuting indeterminates indexed by E. For $B \subseteq E$ let

$\mathbf{y}^B = \prod_{e \in B} y_e$ as before, and define the *basis-enumerator* of $\mathcal{M}$ to be the polynomial

$$B(\mathcal{M}; \mathbf{y}) = \sum_{B \in \mathcal{B}} \mathbf{y}^B.$$

Clearly, if $\mathcal{M} = \mathcal{M}(G)$ is the graphic matroid of a graph G then the basis-enumerator $B(\mathcal{M}; \mathbf{y})$ of $\mathcal{M}$ equals the spanning-tree enumerator $T(G; \mathbf{y})$ of G.

A matroid $\mathcal{M} = (\mathcal{B}, E)$ is a *HPP matroid* if its basis-enumerator $B(\mathcal{M}; \mathbf{y})$ has the half-plane property. A matroid $\mathcal{M} = (\mathcal{B}, E)$ is a *Rayleigh matroid* if its basis-enumerator $B(\mathcal{M}; \mathbf{y})$ satisfies Rayleigh monotonicity. Theorem 13 has the following immediate consequence.

Corollary 14. *Every HPP matroid is a Rayleigh matroid.*

Theorem 9 can be rephrased as saying that every graphic matroid is a HPP matroid. In fact, the proof can be adapted to show much more.

Theorem 15. *Every* $\sqrt[6]{1}$ *matroid is a HPP matroid.*

Proof. Exercise! As a hint, first show that if A is a matrix representing $\mathcal{M}$ as in the definition of a CTU matroid, and if $Y = \mathrm{diag}(y_e : \ e \in E)$ then

$$B(\mathcal{M}; \mathbf{y}) = \det AYA^{\dagger}.$$

□

In summary, we have the hierarchy of classes of matroids indicated in Figure 5. (Smaller classes are at the bottom – larger classes are at the top.) Each of these implications is strict – no two of these properties are equivalent. There are HPP matroids that are not representable over any field [19]. It is natural to wonder whether there is an example of a homogeneous multiaffine polynomial with the half-plane property that is **not** a (weighted) basis-enumerator of a matroid. The answer is NO, and this is proved in [6], but even more is true.

Theorem 16 (Wagner [18], Corollary 4.9.)**.** *Let* $Z(\mathbf{y}) = \sum_{S \subseteq E} c(S)\mathbf{y}^S$ *be a homogeneous multiaffine polynomial with nonnegative real coefficients. If* $Z(\mathbf{y})$ *satisfies Rayleigh monotonicity then*

$$\mathrm{Supp}(Z) = \{S \subseteq E : \ c(S) > 0\}$$

is the set of bases of a matroid.

(The proof of Theorem 16 is gruesome and dull, and is omitted.)

$$\begin{array}{c}\text{Rayleigh}\\ \Uparrow\\ \text{HPP}\\ \Uparrow\\ \sqrt[6]{1}\\ \Uparrow\\ \text{regular}\\ \Uparrow\\ \text{graphic}\end{array}$$

FIGURE 5. A hierarchy of matroid classes.

Thus, the theory of the half-plane property and Rayleigh monotonicity, abstracted away from the motivating examples of electrical networks, finds its natural context as a topic in matroid structure theory.

To conclude this section, we consider structural consequences that follow from a matroid $\mathcal{M}$ being Rayleigh, or HPP.

Proposition 17. *Let $(\mathcal{M}, E)$ be a matroid. The following are equivalent:*
(a) *The basis-enumerator $B(\mathcal{M}; \mathbf{y})$ satisfies Rayleigh monotonicity.*
(b) *Fix $y_c > 0$ for all $c \in E$, and choose a basis B of $\mathcal{M}$ randomly with probability proportional to $\mathbf{y}^B$. Then for any two elements $e, f \in E$,*

$$\Pr[e \in B \text{ and } f \in B] \leq \Pr[e \in B] \cdot \Pr[f \in B].$$

Proof. Let $M = M(\mathbf{y}) = B(\mathcal{M}; \mathbf{y})$, to simplify the notation. The probability of choosing the basis $B \in \mathcal{B}$ is $\mathbf{y}^B/M$. Thus, the probability of choosing a basis B that contains e is

$$\Pr[e \in B] = \frac{y_e M_e}{M}.$$

Similarly, the probability of choosing a basis B that contains f is

$$\Pr[e \in B] = \frac{y_f M_f}{M},$$

and the probability of choosing a basis B that contains both e and f is

$$\Pr[e \in B \text{ and } f \in B] = \frac{y_e y_f M_{ef}}{M}.$$

The inequality in part (b) is thus

$$\frac{y_e y_f M_{ef}}{M} \leq \frac{y_e M_e}{M} \cdot \frac{y_f M_f}{M}.$$

Since all $\mathbf{y} > \mathbf{0}$, this is equivalent to

$$M_{ef} \cdot M \leq M_e \cdot M_f.$$

Now the relations $M_e = M_e^f + y_f M_{ef}$ and $M_f = M_f^e + y_e M_{ef}$ and

$$M = M^{ef} + y_e M_e^f + y_f M_f^e + y_e y_f M_{ef}$$

and some cancellation shows that the inequality in part (b) is equivalent to $\Delta M\{e, f\} \geq 0$. It follows that parts (a) and (b) of the proposition are equivalent. □

The inequalities of Theorem 18 were proven for regular matroids by Stanley [15] in 1981, and the statement about the roots of the polynomial was proved for regular matroids by Godsil [9] in 1982.

Theorem 18 (Choe-Wagner [7], Theorem 4.5.)**.** *Let* $(\mathcal{M}, E)$ *be a HPP matroid of rank* r*, and let* $S \subseteq E$*. For every* $0 \leq k \leq r$*, let* N_k *be the number of bases* B *of* $\mathcal{M}$ *such that* $|B \cap S| = k$*. Then the polynomial* $N_0 + N_1 x + \cdots + N_r x^r$ *has only real nonpositive roots. Thus, if the degree of this polynomial is* d *then for all* $1 \leq k \leq d-1$*,*

$$\frac{N_k^2}{\binom{d}{k}^2} \geq \frac{N_{k-1}}{\binom{d}{k-1}} \cdot \frac{N_{k+1}}{\binom{d}{k+1}}.$$

Proof. Let $M = M(\mathbf{y}) = B(\mathcal{M}; \mathbf{y})$ be the basis-enumerator of $\mathcal{M}$. For each $e \in E$ let $y_e = a_e x + b_e$, in which $a_e = 1$ if $e \in S$ and $a_e = 0$ if $e \in E \smallsetminus S$, and $b_e = 1 - a_e$. Since M has the half-plane property, $M(\mathbf{a}x + \mathbf{b})$ has only real nonpositive roots, by Proposition 11. This is the polynomial in the statement of the theorem. The inequalities follow from the Lemma below. □

Lemma 19 (Newton's Inequalities)**.** *Let* $F(x) = N_0 + N_1 x + \cdot + N_d x^d$ *be a polynomial of degree* d *in* $\mathbb{R}[x]$ *that has only real roots. Then for all* $1 \leq k \leq d-1$*,*

$$\frac{N_k^2}{\binom{d}{k}^2} \geq \frac{N_{k-1}}{\binom{d}{k-1}} \cdot \frac{N_{k+1}}{\binom{d}{k+1}}.$$

Sketch of Proof. It suffices to prove these inequalities when all the roots of $F(x)$ are simple roots, since an arbitrary polynomial with only real roots can be obtained as a limit of such polynomials with only simple roots. Rolle's Theorem implies that if $F(x)$ has only real simple roots, then its derivative $F'(x)$ also has only real simple roots.

Consider any $1 \leq k \leq d-1$. Let $G(x) = (\mathrm{d}/\mathrm{d}x)^{k-1}F(x)$, let $H(x) = x^{d-k+1}G(1/x)$, and let $J(x) = (\mathrm{d}/\mathrm{d}x)^{d-k-1}G(x)$. Then $J(x)$ is a quadratic polynomial with only real simple roots, so the discriminant of this polynomial is positive. This yields Newton's Inequalities above. □

Even for graphs, the conclusion of Theorem 18 can be interesting. As one simple example, let $G = (V, E)$ be a connected graph, let $v \in V$ be a vertex of degree d in G, and for $0 \leq k \leq d$ let N_k be the number of spanning trees T of G such that v has degree k in T. Theorem 18 implies that the polynomial

$$N_0 + N_1 x + \cdots + N_d x^d$$

has only real nonpositive roots, and the resulting inequalities are not obviously obvious.

IV. Random Walks and Electrical Networks.

The system of equations used in the derivation of Kirchhoff's Formula has another interpretation, in terms of "hitting probabilities" for a random walk on the weighted graph $(G, \mathbf{y})$. This leads to probabilistic interpretations of various facts about electrical networks.

Let $G = (V, E)$ be a connected graph, with indeterminate admittances $\mathbf{y} = \{y_e : \ e \in E\}$ on the edges. Orient G arbitrarily, let D be the corresponding signed incidence matrix, and let $L = DYD^\dagger$ be the weighted Laplacian matrix of G. (Here $Y = \mathrm{diag}(y_e : \ e \in E)$ is the diagonal matrix of admittances.) Note that L does not depend on the choice of orientation used to define D.

Let $\boldsymbol{\varphi}$ be a column vector indexed by V, and for $v \in V$ let $\boldsymbol{\delta}_v$ be the V-indexed column vector defined by

$$(\boldsymbol{\delta}_v)_w = \begin{cases} 1 & \text{if } w = v, \\ 0 & \text{if } w \neq v. \end{cases}$$

The system of equations used to derive Kirchhoff's Formula for $\mathcal{Y}_{ab}(G; \mathbf{y})$ is

$$L\boldsymbol{\varphi} = \boldsymbol{\delta}_b - \boldsymbol{\delta}_a.$$

We normalized $\varphi(a) = 0$ and computed $\varphi(b)$ by Cramer's Rule. Alternatively, we could have normalized $\varphi(a) = 0$ and $\varphi(b) = 1$, and calculated the value $\mathcal{Y}$ satisfying

$$L\boldsymbol{\varphi} = \mathcal{Y}\left[\boldsymbol{\delta}_b - \boldsymbol{\delta}_a\right].$$

In this form, the problem can be generalized to that of finding vectors $\boldsymbol{\varphi}$ and $\mathbf{h}$ such that $L\boldsymbol{\varphi} = \mathbf{h}$. In this equation, $\varphi(v)$ is the electrical potential of the vertex v, and $h(v)$ is the external (driving) current being supplied to the vertex v. Proposition 20 is one natural situation in which this problem has a unique solution.

Proposition 20. *Let $(G, \mathbf{y})$ be an electrical network, and let $L = L(G; \mathbf{y})$ be the weighted Laplacian matrix of $(G, \mathbf{y})$. Let $U \subseteq V$ be a nonempty set of vertices, and let $f(v)$ for each $v \in U$ be fixed scalars. Then there is a unique pair of V-indexed column vectors $\boldsymbol{\varphi}$ and $\mathbf{h}$, such that $\varphi(v) = f(v)$ for all $v \in U$, $h(v) = 0$ for all $v \notin U$, and $L\boldsymbol{\varphi} = \mathbf{h}$.*

Proof. Let $W = V \smallsetminus U$ and let $\boldsymbol{\psi}$ be the W-indexed column vector with entries $\psi(v) = \varphi(v)$ for all $v \in W$. Consider just the equations in $L\boldsymbol{\varphi} = \mathbf{h}$ corresponding to rows in W: that is, $L(U|)\boldsymbol{\varphi} = \mathbf{0}$. Subtract from both sides all the terms in $L(U|)\boldsymbol{\varphi}$ that depend on the scalars

$\{f(v) : v \in U\}$. The result is a system of equations of the form $L(U|U)\boldsymbol{\psi} = \mathbf{g}$ for some W-indexed vector $\mathbf{g}$. (In fact, $\mathbf{g} = -L(U|U]\mathbf{f}$, in which $\mathbf{f}$ is the U-indexed column vector with entries $f(v)$ for all $v \in U$.) By Theorem 6, $L(U|U)$ is invertible, so this system has a unique solution for $\boldsymbol{\psi}$. This determines the vector $\boldsymbol{\varphi}$ uniquely, which determines $\mathbf{h} = L\boldsymbol{\varphi}$ uniquely. □

Proposition 20 says that if we fix the electrical potentials of some nonempty subset U of vertices, and if we require that Kirchhoff's Current Law holds at all vertices that are **not** in U, then the currents internal to the network have exactly one solution satisfying Ohm's Law and Kirchhoff's Voltage Law. The external (driving) currents $\mathbf{h}$ are just what is needed to hold all the vertices $v \in U$ at their required potentials $f(v)$.

The problem solved by Proposition 20 is known as the *Dirichlet Problem* for the network $(G, \mathbf{y})$. The vertices in U are on the *boundary*, and the values $\{f(v) : v \in U\}$ are *boundary conditions*. The vertices in $W = V \smallsetminus U$ are in the *interior*, and the function $\boldsymbol{\varphi}$ is said to be *harmonic* at a vertex $v \in V$ if $(L\boldsymbol{\varphi})(v) = 0$. Proposition 20 states that for any network $(G, \mathbf{y})$ and any nonempty set U and any boundary conditions on U, there is a unique function $\boldsymbol{\varphi}$ agreeing with these conditions on the boundary and harmonic at every vertex in the interior.

Dirichlet problems on graphs are discrete analogues of the Dirichlet problems that arise in the theory of partial differential equations (PDEs). Biggs [2] and Doyle and Snell [8] contain more information about Dirichlet problems on graphs, and any introductory text on PDEs should have a section on Dirichlet problems in Euclidean space. Any twice-differentiable manifold M has a differential operator Δ called the *Laplacian operator* on M. For Euclidean d-space

$$\Delta = \nabla \cdot \nabla = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$

These Laplacian differential operators are the continuous analogues of the (weighted) Laplacian matrices that we are dealing with. Dirichlet problems are standard fare in physics and applied mathematics. Not only do they describe electrical networks, but also the transport of heat, diffusive dynamics, and other processes. The rest of this section is essentially concerned with a discrete analogue of diffusion processes.

Let $G = (V, E)$ be a finite connected graph, and let $\mathbf{y} = \{y_e : e \in E\}$ be **positive** real weights on the edges of G. We assume that G has no loops. We can assume that G has no parallel edges by replacing all the edges e_1, ..., e_k between v and w with a single edge e of weight $y_e = y_{e_1} + \cdots + y_{e_k}$. Let $L = L(G; \mathbf{y})$ be the weighted Laplacian matrix of $(G, \mathbf{y})$, and for each vertex $v \in V$ let

$$C_v = L_{vv} = \sum_{e \in E \text{ incident with } v} y_e.$$

The *random walk* on the network $(G, \mathbf{y})$ is the following process. A vertex $v(0) \in V$ is chosen at random according to some probability distribution $\boldsymbol{\pi}(0)$ on V. Thereafter, for each $t \in \mathbb{N}$, if $v(t) = v$ then the next vertex $v(t+1) = w$ is chosen at random, subject to $w \neq v$, with probability equal to $p(v \to w) = -L_{vw}/C_v$. Thus, the sequence

$$v(0), v(1), v(2), v(3), ...$$

defines an infinite walk in the graph G, and at each step the probability that this walk moves from v to w (along e) is y_e/C_v.

Let P be the V-by-V *transition matrix* for the random walk on $(G, \mathbf{y})$: the entries of P are given by $P_{vv} = 0$ for all $v \in V$, and if $v \neq w$ then $P_{vw} = p(w \to v)$. The reason for indexing P like this is that if we think of the probability distribution $\boldsymbol{\pi}(0)$ of $v(0)$ as a V-indexed column vector, then the probability distribution of $v(1)$ is

$$\boldsymbol{\pi}(1) = P\boldsymbol{\pi}(0).$$

It follows that for all $t \in \mathbb{N}$, the distribution of $v(t)$ is $P^t\boldsymbol{\pi}(0)$.

From the general theory of Markov chains, there is a unique probability distribution $\boldsymbol{\pi}$ on V such that $P\boldsymbol{\pi} = \boldsymbol{\pi}$. Moreover, if G is not bipartite then for any choice of $\boldsymbol{\pi}(0)$,

$$\lim_{t \to \infty} P^t \boldsymbol{\pi}(0) = \boldsymbol{\pi}.$$

This distribution $\boldsymbol{\pi}$ is called the *stationary distribution* of P. See Doyle and Snell [8] or Lyons and Peres [11] for details.

We can express the transition matrix P in terms of the weighted Laplacian matrix L. For all $v \in V$ we have $P_{vv} = 0$, and for $v \neq w$ in V we have

$$P_{vw} = p(w \to v) = -L_{wv}/C_w = -L_{vw}/C_w.$$

Let C be the diagonal matrix $C = \mathrm{diag}(C_v : v \in V)$. Thus,

$$P = I - LC^{-1}.$$

The equation defining the stationary distribution, $P\boldsymbol{\pi} = \boldsymbol{\pi}$, is thus $(I - LC^{-1})\boldsymbol{\pi} = \boldsymbol{\pi}$, which is equivalent to

$$LC^{-1}\boldsymbol{\pi} = \mathbf{0}.$$

The kernel of the weighted Laplacian matrix is one-dimensional, and is spanned by the all-ones vector. (Exercise! Use Theorem 5.) Therefore, $C^{-1}\boldsymbol{\pi} = (1/z)\mathbf{1}$ for some scalar z. Since $\boldsymbol{\pi} = (1/z)C\mathbf{1}$ is a probability distribution, the sum of its entries is one, so that $z = \sum_{v \in V} C_v$ and $\boldsymbol{\pi}$ is the V-indexed vector with v-th entry $\pi(v) = C_v/z$ for all $v \in V$. That is, the random walk on $(G, \mathbf{y})$ has the property that the fraction of time that it spends at vertex $v \in V$ is proportional to C_v, for all $v \in V$.

Finally, we consider the question of hitting probabilities for the random walk on $(G, \mathbf{y})$. Fix disjoint nonempty subsets $A, B \subset V$ and for any $v \in V$, let $\eta(v)$ denote the probability that the random walk on $(G, \mathbf{y})$ starting from v (*i.e.* from $\boldsymbol{\delta}_v$) hits a vertex in B before it hits a vertex in A. Clearly $\eta(a) = 0$ for all $a \in A$ and $\eta(b) = 1$ for all $b \in B$. Arrange these *hitting probabilities* $\eta(v)$ for all $v \in V$ in a V-indexed column vector $\boldsymbol{\eta}$.

Consider any vertex $v \in V$ that is not in the set $U = A \cup B$, and let $\mathcal{W}$ be a random walk starting from $v(0) = v$. By considering the possibilities for $v(1) = w$ on $\mathcal{W}$, we see that

$$\eta(v) = \sum_{w \text{ adjacent to } v} p(v \to w)\eta(w).$$

If these equations were true for all vertices in V, then that would be the matrix equation $\boldsymbol{\eta}^\dagger = \boldsymbol{\eta}^\dagger P$, or equivalently $\boldsymbol{\eta}^\dagger LC^{-1} = \mathbf{0}^\dagger$, or equivalently $L\boldsymbol{\eta} = \mathbf{0}$ (since $L^\dagger = L$). But in $L\boldsymbol{\eta} = \mathbf{0}$ only the equations corresponding to rows indexed by $v \in W = V \smallsetminus U$ are required: for $v \in U$, the v-th entry of $L\boldsymbol{\eta}$ need not be zero.

Thus, we seek a solution to the system of equations

$$L\boldsymbol{\eta} = \mathbf{h}$$

such that $\eta(a) = 0$ for all $a \in A$ and $\eta(b) = 1$ for all $b \in B$, and such that $h(v) = 0$ for all $v \in V \smallsetminus (A \cup B)$. This is a Dirichlet problem on the network $(G, \mathbf{y})$! The boundary of the network is $U = A \cup B$, and the boundary values for $\boldsymbol{\eta}$ are as given. The interior of the network is $W = V \smallsetminus U$, and $\boldsymbol{\eta}$ is required to be harmonic at all interior vertices. By Proposition 20, this Dirichlet problem has a unique solution $\boldsymbol{\eta}$ and $\mathbf{h}$.

Now, we have already interpreted the unique solution to the Dirichlet problem $L\boldsymbol{\varphi} = \mathbf{h}$ in terms of electrical networks. The equation $\boldsymbol{\varphi} = \boldsymbol{\eta}$

connects the theory of electrical networks with the theory of random walks. If we ground all the vertices $a \in A$ at an electrical potential of zero volts $\varphi(a) = 0$, and we hold all the vertices $b \in B$ at an electrical potential of one volt $\varphi(b) = 1$, then the electrical currents will flow in such a way that the potential of vertex $v \in W$ is $\varphi(v)$ volts. If we start a random walk at v, then the probability that it hits B before it hits A is $\eta(v)$. The equation $\varphi(v) = \eta(v)$ shows that these two seemingly unrelated quantities are identical.

Doyle and Snell [8] and Lyons and Peres [11] take this connection between random walks and electrical networks much further. Unfortunately, I have run out of time and energy, so I'll just have to leave it at that.

References

[1] N. Balabanian and T.A. Bickart, "Electrical Network Theory," Wiley, New York, 1969.

[2] N.L. Biggs, *Algebraic potential theory on graphs,* Bull. London Math. Soc. **29** (1997), 641-682.

[3] P. Brändén, *Polynomials with the half-plane property and matroid theory,* Adv. Math. **216** (2007), 302-320.

[4] R.L. Brooks, C.A.B. Smith, A.H. Stone, and W.T. Tutte, *The dissection of rectangles into squares*, Duke Math. J. **7** (1940), 312-340.

[5] S. Chaiken, *A combinatorial proof of the all minors matrix tree theorem,* SIAM J. Algebraic Discrete Methods **3** (1982), 319-329.

[6] Y.-B. Choe, J.G. Oxley, A.D. Sokal, and D.G. Wagner, *Homogeneous polynomials with the half-plane property,* Adv. Appl. Math. **32** (2004), 88-187.

[7] Y.-B. Choe and D.G. Wagner, *Rayleigh matroids,* Combin. Probab. Comput. **15** (2006), 765-781.

[8] P.G. Doyle and J.L. Snell, "Random Walks and Electric Networks," Carus Math. Monographs **22**, MAA, Washington DC, 1984.

[9] C.D. Godsil, *Real graph polynomials*, in "Progress in graph theory (Waterloo, Ont., 1982)", 281-293, Academic Press, Toronto, 1984.

[10] G. Kirchhoff, *Über die Auflösung der Gleichungen, auf welche man bei der Untersuchungen der linearen Vertheilung galvanischer Ströme geführt wird,* Ann. Phys. Chem. **72** (1847), 497-508.

[11] R. Lyons with Y. Peres, "Probability on Trees and Networks," `http://mypage.iu.edu/~rdlyons/prbtree/prbtree.html`

[12] J.B. O'Toole, *On the solution of the equations obtained from the investigation of the linear distribution of galvanic currents* IRE Trans. Circuit Theory **5** (1958), 4-7.

[13] J.G. Oxley, "Matroid Theory," Oxford U.P., New York, 1992.

[14] J.G. Oxley, *What is a matroid?,* Cubo **5** (2003), 179-218. `http://www.math.lsu.edu/ oxley/survey4.pdf`

[15] R.P. Stanley, *Two combinatorial applications of the Aleksandrov-Fenchel inequalities*, J. Combin. Theory Ser. A **31** (1981), 56-65.

[16] I. Vágó, "Graph theory: Application to the Calculation of Electrical Networks," Elsevier, Amsterdam, 1985.

[17] D.G. Wagner, *Matroid inequalities from electrical network theory,* Electron. J. Combin. **11** (2004/06), #A1.

[18] D.G. Wagner, *Negatively correlated random variables and Mason's conjecture for independent sets in matroids*, Ann. Combin. **12** (2008), 211-239.

[19] D.G. Wagner and Y. Wei, *A criterion for the half-plane property,* Discrete Math. **309** (2009), 1385-1390.

[20] G. Whittle, *On matroids representable over* GF(3) *and other fields,* Trans. Amer. Math. Soc. **349** (1997), 579603.