Calculus 1 Assignment 10 Solutions

Alex Cowan
cowan@math.columbia.edu

1. If $-2 \le f(x) < 3$ for all x, then

$$\int_5^{11} f(x)\,dx \le \int_5^{11} f(x)\,dx < \int_5^{11} 3\,dx.$$

The integrals on the left and right can be evaluated using FTC, or by reasoning geometrically, and are equal to -12 and 18 respectively.

2. Using the chain rule, we have

$$\begin{aligned}\frac{d}{dx}\int_1^{x^2} f(t)\,dt &= \frac{dx^2}{dx}\frac{d}{d(x^2)}\int_1^{x^2} f(t)\,dt \\ &= 2x \cdot \frac{d}{d(x^2)}\int_1^{x^2} f(t)\,dt\end{aligned}$$

FTC tells us that $\frac{d}{du}\int_a^u f(t)\,dt = f(u)$. The case above is where $u = x^2$, so

$$\frac{d}{dx}\int_1^{x^2} f(t)\,dt = 2xf(x^2).$$

3.
a) Set $u = x^2$. Then

$$\begin{aligned}\int x\sin(x^2)\,dx &= \int x\sin u\,\frac{du}{2x} \\ &= \frac{1}{2}\int \sin u\,du \\ &= \frac{1}{2}\cos u + C \\ &= \frac{1}{2}\cos x^2 + C\end{aligned}$$

for any $C \in \mathbb{R}$.

b) Set $u = \log t$. Then

$$\begin{aligned}\int \frac{1}{1+\log t}\,\frac{dt}{t} &= \int \frac{1}{1+u}\,du \\ &= \log|1+u| + C \\ &= \log|1+\log t| + C\end{aligned}$$

where C is any locally constant function. In this case, C is any function which is constant on $0 < t < e^{-1}$ and $e^{-1} < t$, but could take on different values on each of those intervals.

c) The quantity $\log(x)e^{x^3-3}$ is constant with respect to t, so

$$\begin{aligned}\int_{-1}^{\pi} \log(x)e^{x^3-3}\,dt &= \log(x)e^{x^3-3}\int_{-1}^{\pi} 1\,dt \\ &= \log(x)e^{x^3-3}(\pi+1).\end{aligned}$$

d) This integral is improper. Here we should write

$$\int_{-3}^{0} e^{\frac{1}{x}}\frac{dx}{x^2} = \lim_{h\to 0^-}\int_{-3}^{h} e^{\frac{1}{x}}\frac{dx}{x^2}.$$

Now we make the substitution $u = x^{-1}$, giving

$$\begin{aligned}\lim_{h\to 0^-}\int_{-3}^{h} e^{\frac{1}{x}}\frac{dx}{x^2} &= \lim_{h\to 0^-} -\int_{x=-3}^{x=h} e^u\,du \\ &= \lim_{h\to 0^-}(-e^u + C)\Big|_{x=-3}^{x=h} \\ &= \lim_{h\to 0^-} -e^{\frac{1}{x}}\Big|_{x=-3}^{x=h} \\ &= e^{-\frac{1}{3}} - \lim_{h\to 0^-} e^{\frac{1}{h}} \\ &= e^{-\frac{1}{3}} - 0.\end{aligned}$$

e) We could expand $(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$ and integrate term by term if we wanted, but it's less work to set $u = x+1$. Then $du = dx$ and the integral becomes

$$\int_4^5 u^5\,du = \frac{1}{6}(5^6 - 4^6).$$

4. a) The function $\log x$ is undefined at $x = 0$, so we need to make sure in our Riemann sum we don't ask to evaluate at that point. Otherwise, taking the usual Riemann sum with right endpoints works fine, as long as we make sure n is large enough to actually give a good approximation. Picking $n = 1000$ is fine, but picking $n = 10$ is needlessly inaccurate. So, for example,

$$\int_0^1 \log x\,dx \approx \sum_{k=1}^{1000}\frac{1}{1000}\log\left(\frac{k}{1000}\right).$$

b) Here we can pick endpoints that are very large in absolute value as an approximation to an integral from $-\infty$ to ∞. For example,

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx \approx \int_{-10^{10}}^{10^{10}}\frac{1}{1+x^2}\,dx.$$

Then, when we should make sure to pick a large enough n in our Riemann sum for the rectangles to actually be small. If we picked $n < 10^{10}$, for example, then our rectangles would have a width of about 1, and we'd get a terrible approximation. Here if we pick $n = 10^{100}$, then the width of the rectangles will be about 10^{-90}, which will give a good approximation. For this choice of interval and number of rectangles, we get the approximation

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx \approx \sum_{k=1}^{10^{100}}\frac{2\cdot 10^{10}}{10^{100}}\frac{1}{1+\left(-10^{10}+k\frac{2\cdot 10^{10}}{10^{100}}\right)^2}.$$

c) Here if the number of rectangles is not taken to be significantly more than 1000 the approximation won't be good, because the width of the rectangles would then be comparable to the period of the function, so instead of filling out the area under the curves, you'll instead just draw a bunch of rectangles of essentially random height. If we take $n = 10^{10}$ then we won't have any problems, so

$$\int_{-1}^{1} \cos(1000\pi\theta)\, d\theta \approx \sum_{k=1}^{10^{10}} \frac{2}{10^{10}} \cos\left(-1 + k\frac{2}{10^{10}}\right).$$

5*. a) The definition given in class would give

$$\int_0^1 f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right).$$

Since $\frac{k}{n} \in \mathbb{Q}$, this is

$$\begin{aligned} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \cdot 1 \\ &= \lim_{n\to\infty} n\frac{1}{n} \\ &= 1. \end{aligned}$$

b) Using the definition from class,

$$\begin{aligned} \int_0^\alpha f(x)\, dx + \int_\alpha^1 f(x)\, dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\alpha}{n} f\left(k\frac{\alpha}{n}\right) + \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1-\alpha}{n} f\left(\alpha + k\frac{1-\alpha}{n}\right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\alpha}{n} \cdot 0 + \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1-\alpha}{n} \cdot 0 \\ &= 0. \end{aligned}$$

(Well actually the last term of the second Riemann sum doesn't vanish, but the limit is the same).

c) This question is deceptively tricky. Here we found an instance where

$$\int_a^b f(t)\, dt \neq \int_a^c f(t)\, dt + \int_c^b f(t)\, dt.$$

We didn't make any mistakes, and there is no "contradiction", because nothing told us that we should always have equality between the two quantities above. What happened is that we made an unfortunate choice of definition. We would really like for the two quantities above to be equal in all situations, because it makes our lives easier. When we do math, we're in charge of the definitions we consider. If we start considering some mathematical construction, and it ends up not behaving like we want it to, it's completely reasonable to instead look for similar constructions that might accomplish the same thing, but with nicer properties.

d) The issue with what happened above is that, even though on every short interval $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ our function f took on the values 0 and 1 "basically everywhere", by picking exclusively the right endpoint as a representative for the value of f across the entire interval we didn't capture the behavior of f at all. In fact, if we were to fix any systematic way of picking our sample point, it's not too much of a stretch to think that we could then construct a function g that makes this sampling procedure undesirable in the same way that the function f above made the the "always pick the right endpoint" method of sampling undesirable.

Instead, one of many things we could do is to look at the largest and smallest values that our function takes on a specific interval, and make Riemann sums out of those. Given an interval I (for example $\left[\frac{k}{n}, \frac{k+1}{n}\right]$), define $U(I)$ to be the maximum value of f on the interval I, and similarly define $L(I)$ to be the minimum value of f on I. Then we can construct the "upper Riemann sum" U and the "lower Riemann sum" L of f on an interval

$[a, b]$ defined as

$$U := \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} U\left(\left[a + (k-1)\frac{b-a}{n}, a + k\frac{b-a}{n}\right]\right)$$

and

$$L := \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} L\left(\left[a + (k-1)\frac{b-a}{n}, a + k\frac{b-a}{n}\right]\right).$$

We could then say that $\int_a^b f(t)\,dt$ exists and is equal to U and L if and only if $U = L$. This would make the integrals from parts a) and b) not exist, which is preferable to having them be a counterexample to a property that we'd very much like our definition to have. Sometimes in this kind of situation, we instead just accept that our definition only has the nice property we want in specific situations, but here that's problematic, because it's not totally clear when our nice property will hold, and when it doesn't hold, the situation is so unintuitive and reflects what we're trying to do so poorly that it's reasonable to just rule out the possibility all together.