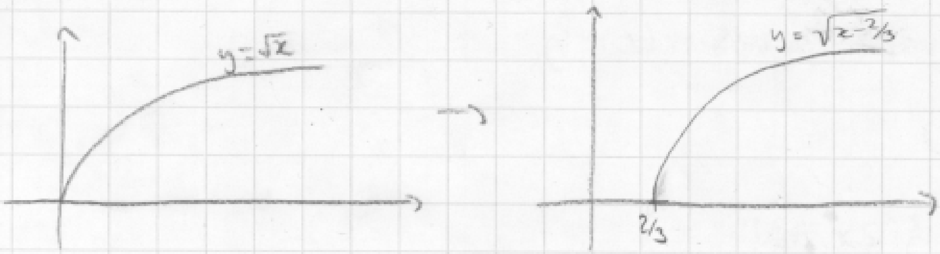
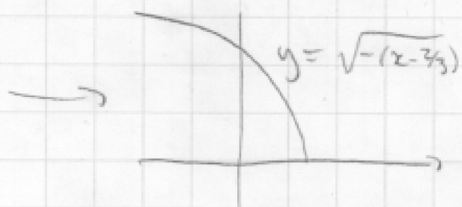
Assignment 2 solutions

1.1 First write $-3x+2 = -3(x-\frac{2}{3})$.

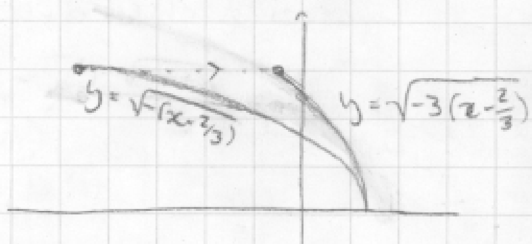
The transformation $x \mapsto x - \frac{2}{3}$ moves everything to the right by $\frac{2}{3}$.



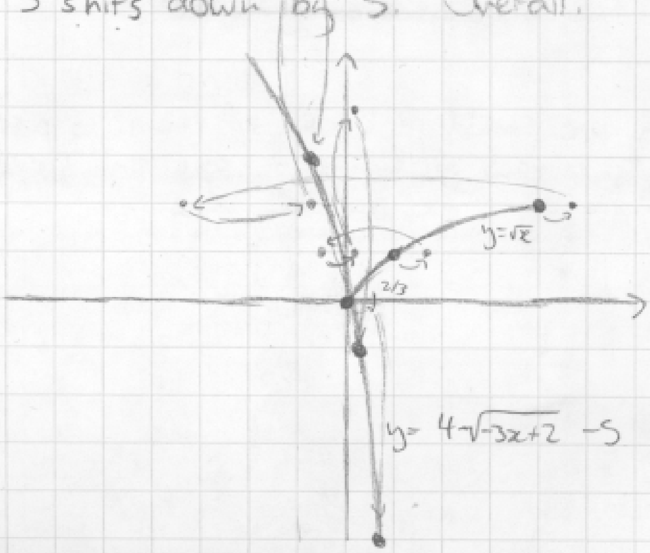
The transformation $x-\frac{2}{3} \mapsto -(x-\frac{2}{3})$ will make the graph go to the lef instead of the right.



$-(x-\frac{2}{3}) \longmapsto -3(x-\frac{2}{3})$ will compress the graph horizontally by a factor of 3.



Multiplying by 4 then stretches horizontally by 4. Finally subtracting 5 shifts down by 5. Overall:



1.2. a) $a(cx+d)+b = acx + ad + b$

b) You could redo the calculation above, or just notice that I'm only relabeling $a \leftrightarrow c$, $b \leftrightarrow d$, giving $g(f(x)) = cax + cb + d$

c) $f(g(x)) = (x^2+1)^2 - 1 = x^4 + 2x^2$

1.3 a) Say $f(x) = \frac{4x-1}{2x+3}$. For any y in the range of f, we want to know what x value solves $f(x) = y$.

$$\frac{4x-1}{2x+3} = y$$

$$4x-1 = y(2x+3)$$

$$4x-1 = 2yx + 3y$$

$$4x - 2yx = 3y - 1$$

$$(4-2y)x = 3y - 1$$

$$x = \frac{3y-1}{4-2y}$$

So, given a y, if we evaluate f at $x = \frac{3y-1}{4-2y}$ we'll get $f(x) = y$.

This means that $f^{-1}(y) = \frac{3y-1}{4-2y}$ (because $f(f^{-1}(y)) = y$, as we just showed)
(we could check directly if we weren't sure)

b) Similarly, we want to solve $x^2 - x = y$ for x.

$$x^2 - x - y = 0$$

Using the quadratic formula,

$$x = \frac{1 \pm \sqrt{1+4y}}{2} \quad \text{(two possibilities for } y > \tfrac{-1}{4}\text{)}$$

To define a function we should pick one of the two possibilities for each $y > -\frac{1}{4}$. I will choose to always pick the $+$ solution. This is equivalent to restricting the domain of $x^2 - x$ to be $x \geq -\frac{1}{2}$.

1.4. To evaluate $f^{-1}(3)$ we want to find an x such that $f(x) = 3$. By inspection $x=1$ works, so $f^{-1}(3) = 1$. It would be hard to give a general form of f^{-1}.

$f(f^{-1}(2)) = 2$, by definition... almost. In class I said f^{-1} is the function with the property that $f^{-1}(f(x)) = x$, and this is backwards. Here's why it's the same.

The domain of f^{-1} is the range of f (stare at the def for a bit), so for any y in $\text{Dom} f^{-1}$, we can find an x such that $f(x) = y$. Then

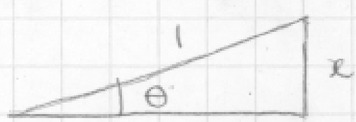
$$f(f^{-1}(y)) = f(f^{-1}(f(x))) \underset{\text{By def of } f^{-1}}{=} f(x) = y$$

So $f(f^{-1}(y)) = y$. You can think of this as $(f^{-1})^{-1} = f$.

You can also reason intuitively: $f^{-1}(2)$ is the number that, when put into f, gives 2. Then you're putting that number into f, so you'll get out 2.

1.5. Set $\theta := \arcsin(x)$. (i.e. just call it something)

Here is a picture that captures what we know:



(We're on the unit circle, so the hypotenuse is 1, the y-coordinate of the top point is x (oops), and the angle is θ. The triangle won't look like this for other values of x or θ, but this helps us reason.)

By pythagoras, the length of this side is $\sqrt{1-x^2}$.

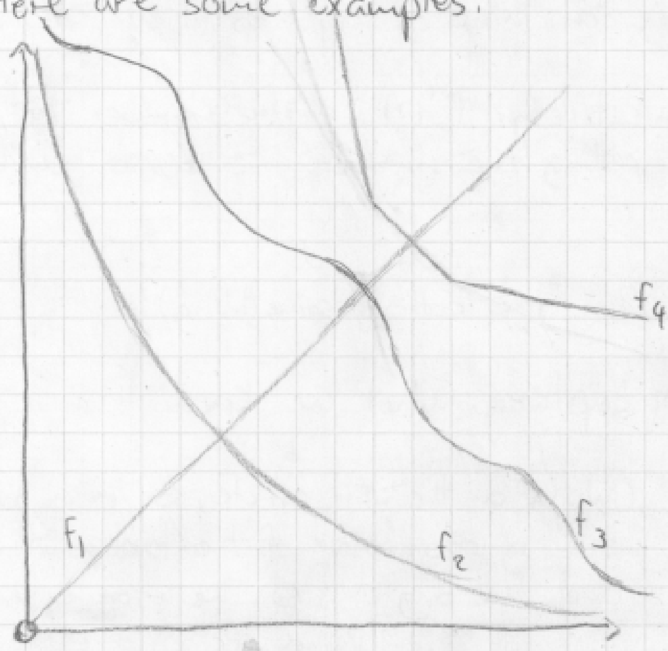
By def, the length of this side is $\cos\theta$.

So $\cos(\arcsin x) = \sqrt{1-x^2}$.

1.6. If $f(f(x)) = x$ for all x, then $f = f^{-1}$.

For any function g, if $g(x) = y$, then $g^{-1}(y) = x$. If we plot the function $g(x)$, then we'll get the points (x, y) where $y = g^{-1}(x)$, or equivalently the points (x,y) where $x = g(y)$. Hence whenever the point (x,y) is in the graph of g, the point (y,x) is in the graph of g^{-1}. Think of x^2 and $\sqrt{x}$. So to produce examples of functions f which satisfy $f(f(x))$, we need graphs that include (y,x) whenever they include (x,y). This is like saying they're symmetric about the line $x = y$ (because reflections across this line swap x and y).

Here are some examples:



3.1 (oops) a) Becaus exp and log are inverses, $x = \exp(\log x)$ and $y = \exp(\log y)$. Then $\log(xy) = \log(e^{\log x} e^{\log y}) = \log e^{\log x + \log y}$. Using the fact that $\log(e^u) = u$, this is $\log x + \log y$.

b) $x = a^{\log_a x}$, so $\log(x) = \log(a^{\log_a x})$. Then, using $\log(u^r) = r \log u$, the right hand side (RHS) is equal to $\log_a x \cdot \log a$. The equation becomes $\log x = \log_a x \cdot \log a$, and the result follows.

3.2 a) If $2^x = 10^3$, then $\log_2(2^x) = \log_2(10^3)$. $\log_2(2^x) = x \log_2(2) = x \cdot 1 = x$. We can also write $x = 3\log 10/\log 2$ using log identities

b) $1 = \log(\log(x)) \Rightarrow e^1 = e^{\log(\log x)} = \log x \Rightarrow e^{e^1} = e^{\log x} = x$, so $x = e^e$

c) Taking the log of both sides, $\log(e^{ax}) = \log(Ce^{bx}) \rightsquigarrow ax \log e = \log C + bx \log e$ $\rightsquigarrow ax = bx + \log C \rightsquigarrow (a-b)x = \log C \rightsquigarrow x = \log C/(a-b)$.