Calculus 1 Assignment 3 solutions

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1.1 The rock hits the ground when $t_1 = 1$. Set $t_0 = 1 - \Delta t$ (this is a short time before the rock hits the ground). In that time, we would like to know how far the rock has fallen. The rock ends up at $h_1 = 0$ (since it's on the ground). We start watching when the rock is at a height $h_0 = h(t_0) = 5 - 5t_0^3$. Writing things in terms of Δt:

$$5 - 5t_0^3 = 5 - 5(1 - \Delta t)^3 = 5 - 5 + 15\Delta t - 15(\Delta t)^2 + 5(\Delta t)^3$$

The rock fell a height $h_0 - h_1 = h_0$ in a time $t_0 - t_1 = -\Delta t$, so our guess for the speed is

$$v(\Delta t) = \frac{15\Delta t - 15(\Delta t)^2 + 5(\Delta t)^3}{-\Delta t} = -15 + 15\Delta t - 5(\Delta t)^2$$

As Δt goes to 0, our guess $v(\Delta t)$ goes to -15, so the rock had velocity -15 when it hit the ground. We get a negative answer because the rock is falling downwards. This is just free information. I would say that the speed (as opposed to the velocity) is 15 (with no negative sign), since for me the speed of something is a non-negative quantity.

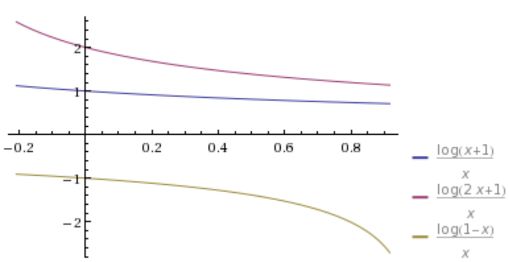
2.1

a) Write $\frac{\sin(\pi x)}{x} = \pi \frac{\sin(\pi x)}{\pi x}$. Then

$$\lim_{x\to 0} \pi \frac{\sin(\pi x)}{\pi x} = \lim_{u \to \frac{0}{\pi}} \pi \frac{\sin(u)}{u} = \pi$$

where we made the substitution $u = \pi x$.

b)Plotting this for $a = 1$, $a = 2$, and $a = -1$ gives the following three graphs:



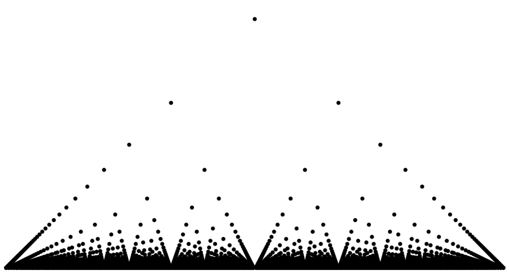
In all three cases, the limit is a, so it's reasonable to guess that it'll be a all the time.

c) $-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$, and both $-|x|$ and $|x|$ go to 0 as x goes to 0. Hence using the squeeze theorem we can conclude that the limit in question is 0 as well.

d) For $x > 0$ we have $\frac{x}{|x|} = 1$, and for $x < 0$ we have $\frac{x}{|x|} = -1$. This means that the right hand limit will be 1, while the left hand limit will be -1. Since these two don't agree, the limit in question doesn't exist.

e) No matter how close we get to 0, the function will be both 1 a lot and 0 a lot. Since we're not tending towards a unique value, this limit doesn't exist. This function is called Dirichlet's Function.

f) Here is a graph of this function on $(0, 1)$:



When we're very close to $x = \frac{1}{2}$, the function has to be small, since all the rational numbers very close to $\frac{1}{2}$ have very large denominators. For example, if you restrict yourself to denominator 5, the closest you can get is $\frac{2}{5}$ or $\frac{3}{5}$, neither of which are very close. If you want denominator 100, the cloest you can get is $\frac{51}{100}$ (without being equal to $\frac{1}{2}$). That's closer, but still not terribly close. Continuing with this pattern, we see that we'll have to take larger and larger denominators to get closer and closer to $\frac{1}{2}$. This means that our function is getting smaller and smaller. Our limit will be 0.

We can prove that the limit is 0 using the squeeze theorem. Clearly $f(x) \geq 0$. Besides $x = \frac{1}{2}$, we can also prove the bound $f(x) \leq 2\left|x - \frac{1}{2}\right|$ (and ignoring $x = \frac{1}{2}$ is fine, since the limit is insensitive to the value of the function at the limit point). Then the squeeze theorem implies that the limit is 0.

Here's how to prove that $f(x) \leq 2\left|x - \frac{1}{2}\right|$. First we calculate $\frac{p}{q} - \frac{1}{2} = \frac{2p-q}{2q}$. This is how far away from $\frac{1}{2}$ we are. We're bounding $f(x)$ by twice this distance, i.e. we're trying to show that

$$\frac{1}{q} \leq 2\left|\frac{2p - q}{2q}\right|$$

Well $2p - q$ can't be 0, since then we would just have $\frac{p}{q} = \frac{1}{2}$, and the value of f at $\frac{1}{2}$ is not relevant. Since it's an integer, it's at least 1 in absolute value. That gives us all we need:

$$2\left|\frac{2p - q}{2q}\right| \geq 2\frac{1}{2q} = \frac{1}{q} = f\left(\frac{p}{q}\right)$$

This function is called Thomae's Function.

2.2

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

is one of many possible solutions. The limit from the left is -1, the limit from the right is 1, and $f(0) = 0$.

2.3 Doing as the hint suggests:

$$\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h \to 0} \frac{f(x+h)g(x+h) + f(x+h)g(x) - f(x+h)g(x) - f(x)g(x)}{h}$$

We can only do this if both $\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$ and $\lim_{h \to 0} \frac{f(x+h)g(x)}{h} - \frac{f(x+h)g(x)}{h}$ exist separately. The latter limit always exist, but the first does not necessarily. That's our first assumption about the niceness of f and g: the limit in the question has to actually exist.

Next group the terms like this:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) + f(x+h)g(x) - f(x+h)g(x) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h}$$

i.e. group the first and third together, and the second and fourth together. We've only done basic arithmetic here, and we can always do that.

Now split up the groups into two separate limits:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h}$$

Here we used the limit law that $\lim u + v = \lim u + \lim v$, and that requires that both limits on the right exist separately. That's our second assumption about the niceness of f and g.

Bring out common factors in each of these limits, like this:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h) - f(x)}{h}$$

This is once again just basic arithmetic, and we can do it all the time.

Here we want to write these limits as a product of two limits:

$$\lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

We're using the limit law $\lim uv = \lim u \lim v$, and we can only do that when each exists separately.

Finally, to get to the expression we're looking for, we want to replace $\lim_{h\to 0} f(x+h)$ with $f(x)$, and $\lim_{h\to 0} g(x)$ with $g(x)$. The second is automatic, but the first requires assuming that f is continuous. Of course for the limits in the previous step to exist, f and g had to have been continuous, but it's okay to be redundant here.