Calculus 1 Assignment 4 solutions

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1. a) Set $f(x) = x^4 + 3x + 1$. Plugging in $x = -1$ and $x = 0$ gives $f(-1) = -1$ and $f(0) = 1$. The function f is continuous on the interval $[0, 1]$ because it's made of elementary functions, and elementary functions are continuous wherever they're defined. Hence f on the interval $[0, 1]$ satisfies the assumptions needed to apply the Intermediate Value Theorem. IVT allows us to conclude that for every y between $f(-1)$ and $f(0)$ there exists an x between -1 and 0 such that $f(x) = y$. If we take $y = 0$, then indeed y is between $f(-1)$ and $f(0)$, so IVT then allows us to conclude that there exists an x in the interval $[-1, 0]$ which satisfies $f(x) = 0$, i.e. f has a real root.

b) We can specify a point on the equator by giving its longitude, which will be a number between 180° W and 0° or between 0° and 180° E. For example the Galápagos Islands lie at about 90° W on the equator. Let's denote the longitude of a location by θ, and let's adopt the convention of writing western longitudes as negative and eastern ones as positive, so that we don't have to specify E or W in the future. Under this convention, the Galápagos Islands lie at $\theta = -90°$, for example.

Given a θ it makes sense to talk about the temperature at that point. Let's define $T(\theta)$ to be the function that, given a θ between $-180°$ and $180°$, returns the temperature at the point on the equator with longitude θ.

In this language, we're trying to prove that there is some θ such that $T(\theta)$ and $T(\theta + 180°)$ are equal (with the understanding that if $\theta + 180°$ exceeds $180°$ then we should subtract $360°$). Define

$$f(\theta) = T(\theta) - T(\theta + 180°).$$

This function is the difference in temperature between a point on the equator and its antipode. The key step is to notice that f satisfies the relationship

$$f(\theta) = -f(\theta + 180°).$$

We can check this directly from the definition, and what might have lead to this observation is noticing that the antipode of the antipode of a point is that point again. For example, the antipode of the Galápagos Islands is at $\theta = 90°$ (which is somewhere in the Indian ocean, roughly equidistant from Sri Lanka, Malaysia, and the Maldives), so

$$\begin{aligned} f(-90°) &= T(-90°) - T(90°) \\ &= -\left(T(90°) - T(-90°)\right) \\ &= -f(90°). \end{aligned}$$

This is saying that

$$T(\text{the Galápagos Islands}) - T(\text{the Maldives}) = -\left(T(\text{the Maldives}) - T(\text{the Galápagos Islands})\right).$$

At the time of writing, $T(-90°) \approx 25°\text{C}$ and $T(90°) \approx 27°\text{C}$, so $f(-90°) \approx -2°\text{C}$ and $f(90°) \approx 2°\text{C}$. If we assume that f is a continuous function, then we can use IVT to conclude that for every temperature difference ΔT between $-2°\text{C}$ and $2°\text{C}$, there exists a θ between $-90°$ and $90°$ for which $f(\theta) = \Delta T$. In particular, there exists a θ for which $f(\theta) = 0$, i.e. $T(\theta) = T(\theta + 180°)$. This proves the assertion the problem asked us to prove. In so doing, we had to assume that f was a continuous function of θ, or equivalently that T was a continuous function of θ (which is a reasonable assumption).

2.
a) $x^2 - 3x - 4 = (x+1)(x-4)$ and $x^2 + 10x + 9 = (x+1)(x+9)$, so the limit in the question is equal to

$$\lim_{x \to -1} \frac{(x+1)(x-4)}{(x+1)(x+9)} = \lim_{x \to -1} \frac{(x-4)}{(x+9)} = -\frac{5}{8}.$$

b) $x^2 - 5x + 6 = (x-2)(x-3)$ and $x^2 - \frac{9}{2}x + 5 = (x-2)(x-5/2)$ (you can get the 5/2 easily by noticing that $(x-a)(x-b) = \text{stuff} + ab$ and that a has to be 2 by the factor theorem from assignment 1), so the limit in question is equal to

$$\lim_{x \to 2} \frac{(x-3)}{(x-5/2)} = 2.$$

c) $x^3 - x^2 + x - 1 = (x-1)(x^2+1)$ and $x^3 - 1 = (x-1)(x^2+x+1)$, so the limit is equal to

$$\lim_{x \to 1} \frac{x^2+1}{x^2+x+1} = \frac{2}{3}.$$

There are many ways to figure out these factorizations (including just staring at them; try it!), but a systematic way I use is to write $x^3 - x^2 + x - 1 = (x-1)(x^2 + ax + b)$ (using the factor theorem) and then expanding and equating the coefficients of the x and x^2 terms.

d) Factor out an x^3 on the top and bottom to get

$$\begin{aligned}
&\lim_{x \to -\infty} \frac{4x^3 + 27x^2 + 1096x - \sin x}{3x^3 + \frac{16}{25}x + x^{-2019}} \\
&= \lim_{x \to -\infty} \frac{x^3}{x^3} \frac{4 + 27x^{-1} + 1096x^{-2} - x^{-3}\sin x}{3 + \frac{16}{25}x^{-2} + x^{-2022}} \\
&= \frac{\lim_{x \to -\infty} 4 \ + \lim_{x \to -\infty} 27x^{-1} \ + \lim_{x \to -\infty} 1096x^{-2} \ - \lim_{x \to -\infty} x^{-3}\sin x}{\lim_{x \to -\infty} 3 \ + \lim_{x \to -\infty} \frac{16}{25}x^{-2} \ + \lim_{x \to -\infty} x^{-2022}} \\
&= \frac{4+0+0-0}{3+0+0} \\
&= \frac{4}{3}.
\end{aligned}$$

Going from the second line to the third line we used a bunch of limit laws, and we can only do that if all the limits exist, which they do because we evaluated all of them in the 4th line. It's important to mention this so that people can follow your work more easily.

e)

$$\begin{aligned}
&\lim_{x \to -\infty} \frac{4x^3 + 27x^2 + 1096x - \sin x}{3x^4 + \frac{16}{25}x + x^{-2019}} \\
&= \lim_{x \to -\infty} \frac{x^3}{x^3} \frac{4 + 27x^{-1} + 1096x^{-2} - x^{-3}\sin x}{3x + \frac{16}{25}x^{-2} + x^{-2022}}
\end{aligned}$$

Here note that the numerator is getting closer and closer to 4, while the denominator is getting very large and negative. If you divide 4 by a very large negative number you get something very close to 0, so the limit is equal to 0.

f)

$$\lim_{x\to-\infty} \frac{4x^5 + 27x^2 + 1096x - \sin x}{3x^4 + \frac{16}{25}x + x^{-2019}}$$
$$= \lim_{x\to-\infty} \frac{x^4}{x^4} \frac{4x + 27x^{-1} + 1096x^{-2} - x^{-3}\sin x}{3 + \frac{16}{25}x^{-2} + x^{-2022}}$$

Here the numerator is getting very large and negative and the denominator is getting closer and closer to 3, so the limit is $-\infty$.

g) e^{-x} gets smaller and smaller as you plug in larger and larger values of x, $\arctan(x)$ gets closer and closer to $\pi/2$, and $\log x$ gets larger and larger, so the function that you're taking a limit of is something getting smaller and smaller divided by something getting bigger and bigger, so the limit is 0.

h) This function is built out of elementary functions, so it's continuous on its domain. We found that 0 was in its domain in assignment 1, so the value of the limit is the value of the function at 0 (this is the definition of continuity), which is $-\cos(-1)/6$.

i) If we set $u = \frac{1}{x}$ then the limit is for $u \to 0$, and we recover the same limit as in h).

j) $\cos x$ doesn't tend towards a unique value as x goes to infinity, so this limit doesn't exist.

3. a) $|x|$ at 0. The limit that appears when you take the derivative is problem 2.1 c) from assignment 3, which doesn't exist.
b) The derivative at x is given by the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

If f is not continuous at x, then $\lim_{h\to 0} f(x+h) \neq f(x)$, so the numerator in this expression is not going to 0. The denominator is going to 0, so (at best) you'll end up dividing some decently-sized number by smaller and smaller numbers, and the limit won't exist. (At worst the top will misbehave horribly and not exist on its own instead of tending to some decently sized number).

4. If we take x to be close to 1, then we can approximate the slope of the tangent line as

$$m \approx \frac{f(x) - f(1)}{x - 1}.$$

This is "the rise over the run". Using the definition of f, this expression becomes

$$m \approx \frac{5 - 5x^2 - (5 - 5 \cdot 1^2)}{x - 1} = \frac{5 - 5x^2}{x - 1}.$$

The slope of the tangent line will be equal to the limit of this expression as x approaches 1. We can evaluate this limit in two ways. The first way is to factor the numerator:

$$5 - 5x^2 = -5(x+1)(x-1),$$

which then gives

$$m = \lim_{x\to 1} \frac{5 - 5x^2}{x - 1} = \lim_{x\to 1} -5(x+1) = -10.$$

The second way is to write $x = 1 + \Delta x$. Then the limit becomes

$$\begin{aligned} m &= \lim_{\Delta x \to 0} \frac{5 - 5(1+\Delta x)^2}{1 + \Delta x - 1} \\ &= \lim_{\Delta x \to 0} \frac{5 - 5(1 + 2\Delta x + (\Delta x)^2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-10\Delta x - 5(\Delta x)^2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} -10 - 5\Delta x \\ &= -10. \end{aligned}$$

The first way is quicker in this situation, but we also got pretty lucky that it worked. I usually end up using the second way more, but it really doesn't matter very much.

We now know the slope of the tangent line, so if we can find a point through which it passes we'll have enough information to determine what it is exactly. The tangent line passes through the point $(1, 0)$, because that's where it's tangent to f. If we can give an equation of a line with slope -10 that passes through the point $(1, 0)$ then we know that that's the tangent line, since a slope and point determine a line uniquely: there is only one line with a given slope that passes through a given point. Notice that the line

$$y = -10(x - 1)$$

has slope -10 (because it's $y = -10x + \text{stuff}$), and it passes through the point $(1, 0)$ (because if you plug in $x = 1$ and $y = 0$ then you solve the equation above, which is what it means for a point to be on a line), so this is the equation of the line tangent to f at the point $(1, 0)$.