Calculus 1 Assignment 6

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Due Wednesday, March 13th at 5 pm

1.
a) What is the derivative of e^t at $t = 0$?
b) What is the derivative of $\arctan x$ at $x = 1$?
c) State the chain rule for $f(g(x))$ in both Newton notation and Leibniz notation.
d) Estimate $\arctan\left(e^{0.1}\right)$. See if you can do it in your head before writing things down (but after that do it while writing things down).

2. For 2.1 through 2.4 below,
a) Estimate the following quantities as best you can without using a calculator,
b) give an estimate of how far off your answer is, and
c) predict if the value you get in a) is an overestimate or an underestimate.

2.1) $\sqrt{2600}$

2.2) $\dfrac{1}{0.0099^2}$

2.3) $\sin(0.01)$

2.4) $\cos(0.01) - 1$

3. Evaluate the following limits.

a) $\displaystyle\lim_{x\to 0} \frac{e^{-x^2} - 1}{\sin^2 x}$

b) $\displaystyle\lim_{x\to 0} \frac{\ln(1-x) - \sin x}{1 - \cos^2 x}$

c) $\displaystyle\lim_{x\to 0^+} \frac{e^{\log x}}{x}$

d) $\displaystyle\lim_{x\to 0^+} x^x$

Hint for d): recall that if f is continuous then $f\left(\lim_{x\to a} g(x)\right) = \lim_{x\to a} f(g(x))$. Can you think of a useful choice of a function f to introduce here?

4.
a) Find the equation of the line tangent to the circle $x^2 + y^2 = 1$ at the point $\left(\frac{3}{5}, -\frac{4}{5}\right)$ without taking any square roots.
b) Find the equation of the line tangent to the curve $y^2 + y = x^3 - x$ at the point $(0, 0)$. (This is my favourite curve. It's an "elliptic curve", and this one is the first to have infinitely many rational solutions.)
c) Plot the curves given by the equations above using WolframAlpha and confirm that your answers make sense. Explain briefly what you were looking for when checking that your answers made sense.

5. A particle is moving along the curve defined by $x^3 + y^3 - 6xy + 3 = 0$. At the time $t = 77$, the particle is at the point $(1, 2)$. At that moment, the x coordinate's rate of change is 1 in the appropriate units (you can take x and y to have units of meters and t to have units of seconds if you like; I personally find it helpful to do this, but for some people this can be confusing or burdensome.)

a) Estimate the particle's x and y coordinates at $t = 77.0001$. Give an order of magnitude estimate of how far off your answer is. (Hint: write the error in terms of big-$\mathcal{O}$ notation, and then hope that the constant involved isn't tiny or huge.)

b) Swap the $\frac{dx}{dt}$ and the $\frac{dy}{dt}$ you found in part a) to get an essentially randomly chosen linear approximation instead of the best one possible. Use this bad approximation to guess x and y coordinates at $t = 77.0001$. Give an order of magnitude estimate of how far off this answer is. By what factor is the error here bigger than the error of the estimate you found in a)?

c) If you plug in the estimates for x and y you got in a) and b) into the expression $x^3 + y^3 - 6xy + 3$ you won't get out exactly 0. Guess the order of magnitude of the number you will get out in both cases, and then check your answer with WolframAlpha.