Calculus 1 Assignment 6 Solutions

Alex Cowan
cowan@math.columbia.edu

1.
a) 1
b) $\frac{1}{2}$
c) In Newton notation, $(f(g(x)))' = f'(g(x))g'(x)$, and in Leibniz notation $\frac{d(f\circ g)}{dx} = \frac{d(f\circ g)}{dg}\frac{dg}{dx} = \frac{df}{dg}\frac{dg}{dx}$.
d) We should do linear approximation here:

$$\arctan\big(e^{0.1}\big) \approx \arctan\big(e^{0}\big) + 0.1 \cdot \frac{d}{dx}\arctan(e^x)\Big|_{x=0}$$

To compute the derivative, we use the chain rule. Using part c) of this problem, we have (in Leibniz notation)

$$\frac{d}{dx}\arctan(e^x)\Big|_{x=0} = \left[\frac{d}{d(e^x)}\arctan(e^x)\Big|_{x=0}\right] \cdot \left[\frac{d}{dx}e^x\Big|_{x=0}\right]$$

We know that $\frac{d}{du}\arctan u = \frac{1}{1+u^2}$ and $\frac{d}{dv}e^v = e^v$, so we get

$$\begin{aligned}\frac{d}{dx}\arctan(e^x)\Big|_{x=0} &= \left[\frac{1}{1+(e^x)^2}\Big|_{x=0}\right] \cdot \left[e^x\Big|_{x=0}\right]\\ &= \left[\frac{1}{2}\right]\cdot[1]\\ &= \frac{1}{2}\end{aligned}$$

We did the calculations for each of the individual factors in parts a) and b) and we determined how they're put together in part c). Now using linear approximation, we have

$$\begin{aligned}\arctan\big(e^{0.1}\big) &\approx \arctan\big(e^{0}\big) + 0.1 \cdot \frac{d}{dx}\arctan(e^x)\Big|_{x=0}\\ &\approx \frac{\pi}{4} + \frac{1}{20}\end{aligned}$$

2.
2.1. Here we can do linear approximation with $f(x) = \sqrt{x}$. To pick our a to expand around, we can choose 2500, because $\sqrt{2500} = 50$, and if we do this we'll get a pretty good result (like we did in class for $\sqrt{26}$, where we took $a = 25$), but we can do significantly better by noticing that $51^2 = (50+1)^2 = 2500 + 100 + 1 = 2601$ and taking $a = 2601$. Doing this, we get

$$\sqrt{2600} \approx \sqrt{2601} + \frac{1}{2\sqrt{2601}}(2600 - 2601) = 51 - \frac{1}{102}$$

We can approximate the error term here with the new term we get if we do a second order approximation instead of a first order approximation. If we were to do a second order approximation, the term we'd add would be

$$\frac{f''(a)}{2}(x-a)^2$$

We compute that $\frac{d^2}{dx^2}\sqrt{x} = \frac{-1}{4x^{\frac{3}{2}}}$, so the new term would be

$$\frac{1}{2}\frac{-1}{4\cdot 2601^{\frac{3}{2}}}(2600-2601)^2 = \frac{-1}{8\cdot 51^3}$$

To get a sense of how big this is, we can guess that $51^3 \approx 50^3 = 10^3 \cdot 125$, so overall our error term is something like -10^{-6}. If you used $a = 2500$ you instead get an error term of about -10^{-2}, which is still very good, but of course this one is much better. Finally, we see that the estimate $51 - \frac{1}{102}$ is an overestimate, because our error term is negative. We could also have reasoned geometrically: the tangent line of $\sqrt{x}$ at $x = 2601$ lies above the curve $y = \sqrt{x}$. Linear approximation is the practice of replacing our function with the tangent line, so we see that if we were to take points on the line instead of on the curve, we'd be guessing y values that are too large.

2.2. Let's take $f(x) = \frac{1}{x^2}$ and $a = 10^{-2}$. Then $f'(x) = \frac{-2}{x^3}$. Using linear approximation,

$$\frac{1}{0.0099^2} \approx \frac{1}{(10^{-2})^2} + \frac{-2}{(10^{-2})^3}\cdot(-10^{-4}) = 10^4 + 2\cdot 10^2$$

To get an error term we again use the new term we'd have in a second order approximation, $\frac{f''(a)}{2}(x-a)^2$. Here we have $f''(a) = \frac{6}{x^4}$, so our guess for the error term is

$$\frac{1}{2}\frac{6}{(10^{-2})^4}(-10^{-4})^2 = 3$$

This is positive so our initial guess was very likely to have been an underestimate. If you ask a calculator you find that $0.0099^{-2} = 10203.04050607080910111213...$, so you can see the contributions from each order of approximation very clearly!

2.3. Using linear approximation we have $\sin x = x$ when x is near 0. Hence we should guess that $\sin(0.01) \approx 0.01$. To estimate an error term, we first try to once again use $\frac{f''(a)}{2}(x-a)^2$. Doing this (with $f(x) = \sin x$ and $a = 0$) gives 0, however, because $\sin''(0) = -\sin(0) = 0$. We can't possibly justify estimating our error as 0, so we should press on and instead use the new term in a third order approximation, which is the most significant term after the ones appearing in the linear approximation. The third order term is of the form $\frac{f'''(a)}{6}(x-a)^3$, and in our situation, that's $\frac{-\cos(0)}{6}\cdot 0.01^3 = -\frac{1}{6}10^{-6}$. This number is negative, so our original guess was likely an overestimate. We could once again also have reasoned geometrically. Checking with a calculator, we have $\sin(0.01) = 0.009999833334...$, so, as expected, this is 0.01 to 6 digits. Moreover note that up until the 4, this is identical to $0.01 - \frac{1}{6}10^{-6}$.

2.4. If we use linear approximation with $f(x) = \cos x - 1$ and $a = 0$, we find that $f(x) \approx f(0) + f'(0)x = 0 + 0x$. In this case, our second order term will be our main term, not our error term. The second order term is $\frac{-\cos(0)}{2}x^2 = -\frac{x^2}{2}$, so we should estimate that $\cos(0.01) - 1 \approx -\frac{1}{2}10^{-4}$. To get an error term, we should first try $\frac{f'''(a)}{6}(x-a)^3$, but here we find $f'''(0) = 0$, so, similarly to problem 2.3, we need to do more work. The next most significant term is $\frac{f''''(a)}{4!}(x-a)^4$, which in this case is $\frac{1}{24}10^{-8}$. This number is positive, so our original guess was an underestimate. Unlike the previous three problems, it's pretty hard to tell if $-\frac{1}{2}10^{-4}$ is an overestimate or an underestimate just by imagining the graph.

3.

a) When u is near 0, we have

$$e^u = 1 + u + \mathcal{O}(u^2)$$

and

$$\sin u = u + \mathcal{O}(u^2)$$

from the linear approximation theorem. For the numerator, we can set $u = -x^2$ to get

$$e^{-x^2} = 1 - x^2 + \mathcal{O}(x^4)$$

For the denominator we have

$$\sin^2 x = (x + \mathcal{O}(x^2))^2 = x^2 + 2x\mathcal{O}(x^2) + \mathcal{O}(x^4) = x^2 + \mathcal{O}(x^3)$$

Thus

$$\begin{aligned} \lim_{x\to 0} \frac{e^{-x^2} - 1}{\sin^2 x} &= \lim_{x\to 0} \frac{1 - x^2 + \mathcal{O}(x^4) - 1}{x^2 + \mathcal{O}(x^3)} \\ &= \lim_{x\to 0} \frac{-1 + \mathcal{O}(x^2)}{1 + \mathcal{O}(x)} \\ &= -1 \end{aligned}$$

b) Using the same approach, we have

$$\begin{aligned} \ln(1-x) &= -x + \mathcal{O}(x^2), \\ \sin(x) &= x + \mathcal{O}(x^2), \text{ and} \\ 1 - \cos^2 x &= x^2 + \mathcal{O}(x^3). \end{aligned}$$

The last of these can be done more quickly by writing $1 - \cos^2 x = \sin^2 x$, but you don't have to do this. All together, we have

$$\lim_{x\to 0} \frac{\ln(1-x) - \sin x}{1 - \cos^2 x} = \lim_{x\to 0} \frac{-x + \mathcal{O}(x^2) - x + \mathcal{O}(x^2)}{x^2 + \mathcal{O}(x^3)} = \lim_{x\to 0} \frac{-1 + \mathcal{O}(x) - 1 + \mathcal{O}(x)}{x + \mathcal{O}(x^2)}.$$

This is basically the limit as x goes to 0 of $\frac{-2}{x}$, so the limit does not exist. This is the limit from the movie Mean Girls.

c) $e^{\log x} = x$, so the limit is just 1. However this problem is great for practicing with L'Hôpital's rule.

d) The hint is leading you instead consider $\log\left(\lim_{x\to 0} x^x\right)$. Because log is continuous, this is equal to

$$\lim_{x\to 0} \log(x^x) = \lim_{x\to 0} x \log x.$$

We can rewrite this as

$$\lim_{x\to 0} \frac{\log x}{x^{-1}}$$

Now we can apply L'Hôpital's rule to deduce that

$$\lim_{x\to 0} \frac{\log x}{x^{-1}} = \lim_{x\to 0} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0} -x = 0.$$

This tells us that $\log\left(\lim_{x\to 0} x^x\right) = 0$. Hence the limit in the question is equal to $e^0 = 1$. There are ways to solve this problem without using L'Hôpital's rule as well.

4.

a) The slope of the tangent line is $\frac{dy}{dx}$, so we need a way to figure out what $\frac{dy}{dx}$ is equal to. We can do this by differentiating the equation $x^2 + y^2 = 1$ with respect to x:

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}1.$$

We have $\frac{d}{dx}x^2 = 2x$, $\frac{d}{dx}1 = 0$, and $\frac{d}{dx}y^2 = \frac{dy}{dx}\frac{d}{dy}y^2 = 2y\frac{dy}{dx}$ using the chain rule. Hence

$$2x + 2y\frac{dy}{dx} = 0.$$

We can then solve for $\frac{dy}{dx}$, obtaining

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Any line through the point $\left(\frac{3}{5}, -\frac{4}{5}\right)$ will be of the form $y = m\left(x - \frac{3}{5}\right) - \frac{4}{5}$, since the given point satisfies this equation. There are other totally fine ways to come to this conclusion if you don't like this kind of reasoning. We found that the slope in this situation should be $-\frac{\frac{3}{5}}{-\frac{4}{5}} = \frac{3}{4}$, so the equation of the tangent line is

$$y = \frac{3}{4}\left(x - \frac{3}{5}\right) - \frac{4}{5}.$$

b) Again, we first need to find the slope of the tangent line, which will be $\frac{dy}{dx}$ at $(0, 0)$. Differentiating the equation $y^2 + y = x^3 - x$ with respect to x, we obtain

$$2y\frac{dy}{dx} + \frac{dy}{dx} = 3x^2 - 1.$$

The simplest way to proceed is to substitute $(x, y) = (0, 0)$ immediately, which turns the equation above into $\frac{dy}{dx} = -1$. We could have solved for $\frac{dy}{dx}$ for general x and y first, and then put in $(x, y) = (0, 0)$ if we wanted to, but this is more work than we need to do.

The tangent line goes through the point $(0, 0)$, so it's necessarily of the form $y = mx$. We found that the slope should be -1, so the equation of the tangent line is $y = -x$.

c) I would look at the curves near where the designated points are and check that the slope of the tangent line on the picture looks like it's similar to what I calculated. It's sort of hard to tell, but you can at least check that it has the right sign and whether or not it's bigger than 1 in absolute value. You can also plot the tangent line on top of the curve, and then it's really clear if you're right or not.

5.

a) The coordinate x is a function of t, and the question is asking us to estimate $x(77.0001)$. We're given that $x(77) = 1$ and $x'(77) = 1$ from the question, where here $x' = \frac{dx}{dt}$ is the rate of change of x with respect to t. We can then use linear approximation to get that $x(77.0001) \approx 1 + 1 \cdot 0.0001 = 1.0001$. The linear approximation theorem tells us that as $\Delta t \to 0$ we have $x(77 + \Delta t) = x(77) + \Delta t + \mathcal{O}\big((\Delta t)^2\big)$, and 0.0001 is a pretty small Δt. If we guess that the implied constant in the big-$\mathcal{O}$ notation is not too big or too small then we could guess that the estimate $x(77.0001) \approx 1.0001$ is off by about $0.0001^2 = 10^{-8}$. Here "about" means maybe to within an order of magnitude or two. This is sort of like plugging in $h = 0.0001$ in a limit where h is going to 0; sometimes it gets you pretty close, but you can't actually be sure. In fact, if $x'(77) = 5$ then the error ends up being about 10^{-6}, not 10^{-8}, so while this often works, it often doesn't as well. Still, this is something people do and it's useful.

We'd like to do the same for y, i.e. approximate $y(77.0001) \approx y(77) + y'(77) \cdot 0.0001$, and while we know from the statement that $y(77) = 2$, we don't know $y'(77)$. Here again y' represents the derivative of y with respect to t, and not x. To find y' we can use the fact that at all times t, the coordinates x and y satisfy the equation $x^3 + y^3 - 6xy + 3 = 0$. Differentiating both sides of this equation with respect to t gives

$$3x^2 \frac{dx}{dt} + 3y^2 \frac{dy}{dt} - 6x \frac{dy}{dt} - 6y \frac{dx}{dt} = 0.$$

Then we can substitute $x = 1$, $y = 2$, and $\frac{dx}{dt} = 1$ to find that at $t = 77$ we have

$$3 + 12 \frac{dy}{dt} - 6 \frac{dy}{dt} - 12 = 0.$$

From this we obtain $y'(77) = 1.5$. We could also have differentiated the equation of the curve with respect to x instead of t to find $\frac{dy}{dx}$ and then used $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$. However do note that we are looking specifically for $\frac{dy}{dt}$ and not $\frac{dy}{dx}$ here.

Once we have $y'(77) = 1.5$ we can do linear approximation to get $y(77.0001) \approx 2.00015$, and the error term will once again be about 10^{-8}, probably.

b) If we pretend that $x'(77) = 1.5$ and $y'(77) = 1$, then "linear approximation" would give $x(77.0001) \approx 1.00015$ and $y(77.0001) \approx 2.0001$. From a) we know that $x = 1.0001$ and $y = 2.00015$ is correct to about 8 digits, so we expect these to start being wrong around the 4^{th} or 5^{th} digit. This means that this "linear approximation" really doesn't help at all. What going on is that you took your 0^{th} order approximation, which was $x = 1, y = 2$, and you added 0.0001 times some random number, so you expect to be off by about 0.0001. The linear approximation theorem doesn't apply here, because the slope of your line is wrong.

c) We know that $x(77.0001) \approx 1.0001 \pm 10^{-8}$ and $y(77.0001) \approx 2.00015 \pm 10^{-8}$. Therefore

$$1.0001 \approx x(77.0001) \pm 10^{-8} \text{ and } 2.00015 \approx y(77.0001) \pm 10^{-8}.$$

Here and henceforth when I write 10^{-8} I just mean some quantity about that big. Plugging in 1.0001 and 2.00015 into the equation of the curve then gives

$$\begin{aligned}
&1.0001^3 + 2.00015^3 - 6 \cdot 1.0001 \cdot 2.00015 + 3 \\
&= (x(77.0001) \pm 10^{-8})^3 + (y(77.0001) \pm 10^{-8})^3 - 6(x(77.0001) \pm 10^{-8})(y(77.0001) \pm 10^{-8}) + 3 \\
&= x(77.0001)^3 \pm 3x(77.0001)^2 10^{-8} + y(77.0001)^3 \pm 3y(77.0001)^2 10^{-8} \\
&\qquad - 6x(77.0001)y(77.0001) \pm 6(x(77.0001) + y(77.0001))10^{-8} + 3 \\
&= x(77.0001)^3 + y(77.0001)^3 - 6x(77.0001)y(77.0001) + 3 \pm 10^{-8} \\
&= \pm 10^{-8}.
\end{aligned}$$

In the third line we dropped all the terms that were a product of two terms of size 10^{-8}, because those terms will be of size 10^{-16} and can just be absorbed into the errors of size 10^{-8}. In the fourth line we combined a sum of a bunch of terms of size 10^{-8} into a single term of size about 10^{-8}.

Similarly using the estimate in b) we expect to not get 0 from the equation of the curve, but something about the size of 10^{-4}. The steps are the same, but just with 10^{-4} everywhere instead of 10^{-8}.