Calculus 1 Assignment 7 Solutions

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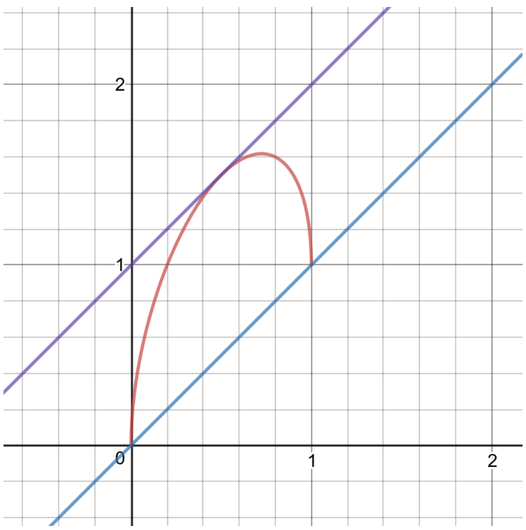
1. There are many possibilities, of course. Here is one:



2. It's impossible for f to be differentiable at 0 and 1. The derivative is a limit, and for a limit to exist, both one-sided limits need to exist. To the left of 0 and to the right of 1 the function f is not defined, so these limits can't exist.

3. Here are some:
a)

$$f(x) = \begin{cases} 0, & x = 0 \\ 1, & 0 < x \leq 1 \end{cases}$$

b)

$$f(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1 \end{cases}$$

4. The function $f(x) = |x - \frac{1}{2}|$ is an example.

5.
a) We have $\frac{d}{dx} \sin x = 0$ if and only if $x = \frac{\pi}{2} + n\pi$ for some n in $\mathbb{Z}$. These are the only points where $\sin x$ can have an extremum, because everywhere else the derivative is defined and nonzero. When $x = \frac{\pi}{2} + 2n\pi$ for some $n \in \mathbb{Z}$ we have a local maximum, because $\cos x$ goes from positive to negative at those points, so $\sin x$ goes from increasing to decreasing, which happens when you're at a maximum. Similarly when $x = \frac{3\pi}{2} + 2n\pi$ for some $n \in \mathbb{Z}$ we have a local minimum. The values of $\sin x$ at these points are all 1 or -1, so all of the local maxima

are additionally global maxima, and all the local minima are also global minima.

b) Here we still have the local and global maximum of $\left(\frac{\pi}{2}, 1\right)$. There are no other points in the domain where the derivative is 0 or does not exist, so this is the only extremum. The intuition for the absense of minima is that, as soon as you pick some point in the domain that you think might be a minimum, you can immediately find another point that's smaller. For example, $\left(\frac{\pi}{6} + 0.01, \sin\left(\frac{\pi}{6} + 0.01\right)\right)$ is not a local minimum, because $\sin\left(\frac{\pi}{6} + 0.01\right) > \sin\left(\frac{\pi}{6} + 0.001\right)$. The endpoints are not in the domain of the function, so these can't be minimums either, because the function isn't defined there. It's important to read the definition carefully and take everything in that definition literally.

c) The function $|\sin x|$ for $x \in [-4, 4]$ is equal to the function

$$\begin{cases} \sin x, & x \in [-4, -\pi] \cup [0, \pi] \\ -\sin x, & x \in (-\pi, 0) \cup (\pi, 4]. \end{cases}$$

(We can figure out how many periods we have by remembering that $\pi \approx 3$.) Thus the derivative of this function is

$$\frac{d}{dx}|\sin x| = \begin{cases} \cos x, & x \in (-4, -\pi) \cup (0, \pi) \\ -\cos x, & x \in (-\pi, 0) \cup (\pi, 4). \end{cases}$$

This derivative is 0 when $x = \frac{-\pi}{2}$ and $\frac{\pi}{2}$, and doesn't exist when $x = -4, -\pi, 0\pi$, and 4. The value of the function at these points are $1, 1, \sin 4, 0, 0, 0$, and $\sin 4$ respectively. This tells us that $\left(\frac{-\pi}{2}, 1\right)$ and $\left(\frac{\pi}{2}, 1\right)$ are local and global maxima, while $(-\pi, 0), (0, 0)$, and $(\pi, 0)$ are local and global minima. At $x = -3.99$ the derivative of the function is negative (and we know that it doesn't change sign between -3.99 and -4), so $-4, \sin 4$ is a local maximum, and the function is even, so $4, \sin 4$ is also a local maximum.

d) The derivative of this function is $12x^3 + 12x^2$, which is 0 when $x = 0$ or $x = -1$. The derivative of this function isn't defined when $x = -2$, so these are the only points where we might have extrema. Between $x = -2$ and $x = -1$ the derivative is negative, so $(-2, 16)$ is a local maximum. Between $x = -1$ and $x = 0$ and between $x = 0$ and $x = 5$ the derivative is positive, which means that $(-1, -1)$ is a local min, and that $(0, 0)$ is neither a local max nor a local min. We can also conclude that $(-1, -1)$ is a global min, because the function is decreasing everywhere to the left and increasing everywhere to the right. There is no global max, because the only local maximum we found was $(-2, 16)$, and when $x = 4.99$, for example, we have $3x^4 + 4x^3 > 16$.

e) The derivative of $\frac{1}{x}$ exists and is nonzero on its entire domain, so this function has no maxima or minima.

6. Call the two numbers x and y. We're looking to minimize the quantity xy subject to the constraint $x - y = 100$. We can use the constraint to write $x = 100 + y$. Then the quantity we're trying to minimize becomes $y(100 + y)$, where y can be any real number. The derivative of $y(100 + y)$ is $100 + 2y$, which exists everywhere and is 0 only when $y = -50$. If $y = -50$, then $x = 50$, and the product is -2500. This is a minimum, because both $y = -51$ and $y = -49$ give larger products, for example.

7. Here we are trying to minimize $x + y$ subject to the constrants $xy = 100$ and $x, y > 0$. We can use the first constraint to write $y = \frac{100}{x}$, and then the second constraint is just $x > 0$ (since then $\frac{100}{x}$ will also be positive). So we are trying to minimize $x + \frac{100}{x}$ for $x > 0$. The derivative of this function is $1 - \frac{100}{x^2}$, which exists everyone on the domain and is 0 when $x = 10$. The sign of the derivative $1 - \frac{100}{x^2}$ is negative when x is less than 10 and positive when x is greater than 10, so this is a minimum. The sum we obtain is 20.