Calculus 1 Assignment 8 solutions

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1.
a) $2x^3 - 3x^2 - 6x + 7$

It's not always possible to figure out where a function will be positive and negative. This is usually the hardest part of sketching, and sometimes you have to just proceed without this information. Here, however, it is possible to figure this out. The function $f(x) = 2x^3 - 3x^2 - 6x + 7$ is continuous on $\mathbb{R}$, so it can only switch signs at its roots. Finding the roots of a cubic polynomial in general is hard, but here we're lucky: $x = 1$ is a root. You can evaluate directly that $f(1) = 0$, but how might you guess this? In general it's reasonable to plug in a few small, simple numbers and see if you get lucky. For polynomials, there's a theorem called the "Rational Roots Theorem" that tells you that if $\frac{p}{q}$ is a rational number in lowest terms that's a root of a polynomial, then p has to divide the constant term of that polynomial, and q has to divide the leading coefficient of that polynomial. Here that means that the only rational numbers you have to test for roots are $\pm 1, \pm 7, \pm \frac{1}{2}$, and $\pm \frac{7}{2}$.

Once you know that $x = 1$ is a root of f, you can write $2x^3 - 3x^2 - 6x + 7 = (x-1)(ax^2 + bx + c)$ using the theorem from assignment 1, problem 2.2. Then, equating coefficients, you quickly find that $a = 2, c = -7$, and $b = -1$. The roots of $2x^2 - x - 7$ are $\frac{1 \pm \sqrt{57}}{4}$.

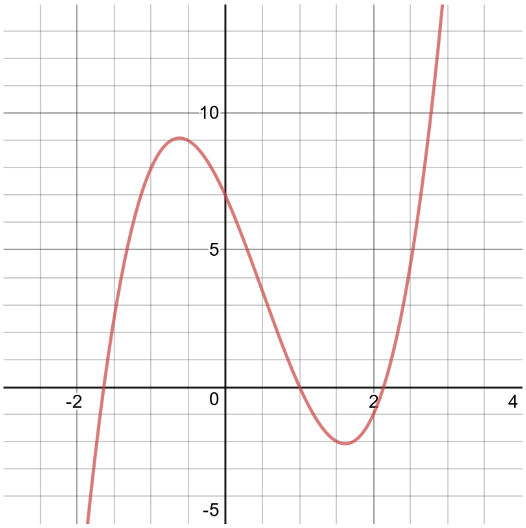
We know that f won't change sign except possibly at these three roots, so we can determine the sign of f by simply evaluating it at any point between the roots. Since $f(100000) > 0$, we know that for all $x > \frac{1+\sqrt{57}}{4}$, we'll have $f(x) > 0$. Similarly, $f(x) < 0$ for all $x < \frac{1-\sqrt{57}}{4}$ and for all x satisfying $1 < x < \frac{1+\sqrt{57}}{4}$, and $f(x) > 0$ when $\frac{1-\sqrt{57}}{4} < x < 1$.

Next we compute that $f'(x) = 6(x^2 - x - 1)$, which has roots at $x = \frac{1 \pm \sqrt{5}}{2}$. Again, f' is continuous on $\mathbb{R}$, so it can only change sign at its roots, so evaluating $f(10000), f(0)$, and $f(-100000)$ we can conclude that $f'(x) > 0$ when $x > \frac{1+\sqrt{5}}{2}$ or $x < \frac{1-\sqrt{5}}{2}$ and $f'(x) < 0$ when $\frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}$.

Then we compute that $f''(x) = 6(2x - 1)$, which is positive when $x > \frac{1}{2}$ and negative when $x < \frac{1}{2}$.

To sketch this graph, we want to sketch a third degree polynomial. This polynomial should be positive when $\frac{1-\sqrt{57}}{4} < x < 1$ or when $x > \frac{1+\sqrt{57}}{4}$, and negative otherwise (except at the roots of course). The polynomial should be decreasing when $\frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}$ and increasing otherwise, and it should be concave up when $x > \frac{1}{2}$ and concave down otherwise.

It's helpful at this point to note that $\sqrt{5}$ is a bit more than 2 and that $\sqrt{57}$ is roughly halfway between 7 and 8, so $\frac{1+\sqrt{57}}{4} \approx 2.1$, $\frac{1-\sqrt{57}}{4} \approx -1.6$, $\frac{1+\sqrt{5}}{2} \approx 1.6$, and $\frac{1-\sqrt{5}}{2} \approx -0.6$. Then you can sketch this graph:



b) $e^{-x} \cos x$ for $x \geq 0$

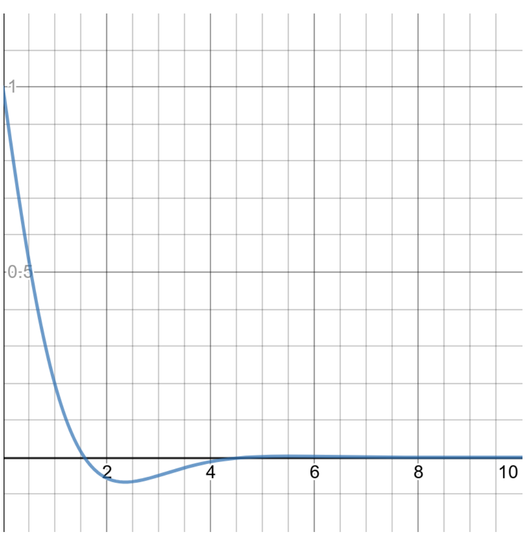
Set $f(x) = e^{-x} \cos x$. We'll need to know that $f'(x) = -e^{-x}(\cos x + \sin x)$ and $f''(x) = 2e^{-x} \sin x$.

Our function $f(x)$ is positive exactly when $\cos x$ is positive, which is when $0 \leq x < \frac{\pi}{2}$, or $3\frac{\pi}{2} < x < 5\frac{\pi}{2}$, or $7\frac{\pi}{2} < x < 9\frac{\pi}{2}$, or $11\frac{\pi}{2} < x < 13\frac{\pi}{2}$, and so on, and it's negative elsewhere, except for the roots.

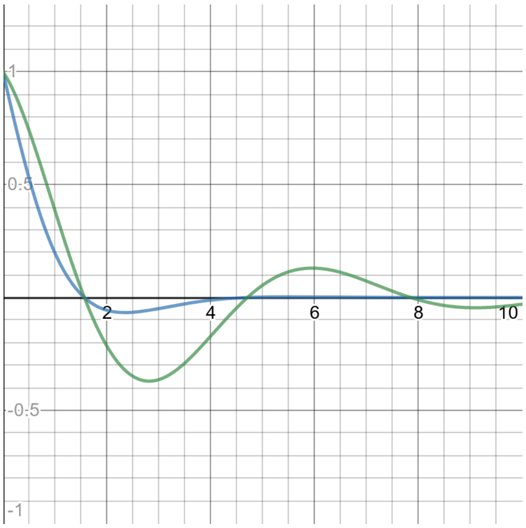
The derivative $f'(x)$ is positive exactly when $\cos x + \sin x < 0$. We know that $\cos x = -\sin x$ when $x = \frac{3\pi}{4} + n\pi$ for some $n \in \mathbb{Z}$, and these functions are continuous on $\mathbb{R}$, so we know the sign doesn't change except possibly at these points. Then we can check directly that $f'(x) > 0$ when $-\frac{\pi}{4} + n\pi < x < \frac{3\pi}{4} + n\pi$ (and $x > 0$, since otherwise the function isn't defined).

The second derivative $f''(x)$ is positive when $\sin x > 0$, which is when $0 < x < \pi$ or $2\pi < x < 3\pi$ or $4\pi < x < 5\pi$, etc.

We now note that we have a horizontal asymptote at $y = 0$. Because e^{-x} decays so quickly, the actual graph doesn't look like much:



Here you can see what these kinds of functions look like when you make the decay less extreme, but you'll notice that this moves around the extrema and inflection points:



c) $\frac{x^3}{x^2-4}$

Set $f(x) = \frac{x^3}{x^2-4}$. Here the roots are easy: $x = 0$ is the only one. However, f can also change sign at $x = \pm 2$ since it has asymptotes there. Between $-\infty$ and -2 we see that $f(x)$ is negative, since at $x = -10000$ the numerator is negative and the denominator is positive. Between $-2 and 0$ we see that $f(x)$ is positive because $f(-1) = \frac{1}{3}$. Between 0 and 2 we see that $f(x)$ is negative because $f(1) = \frac{-1}{3}$, and between 2 and ∞ we see that $f(x)$ is positive because $f(100000) > 0$.

Differentiating $f(x)$ can be laborious, but no matter how you do it you'll get there. I think the easiest way to proceed is to write

$$\frac{x^3}{x^2-4} = x\frac{x^2-4+4}{x^2-4} = x\left(1 + \frac{4}{x^2-4}\right).$$

Then

$$\begin{aligned} f'(x) =& 1 + \frac{4}{x^2-4} - \frac{8x^2}{(x^2-4)^2} \\ =& \frac{(x^2-4)^2 + 4(x^2-4) - 8x^2}{(x^2-4)^2} \\ =& \frac{(x^4 - 8x^2 + 16) + (4x^2 - 16) - 8x^2}{(x^2-4)^2} \\ =& \frac{x^4 - 12x^2}{(x^2-4)^2} \end{aligned}$$

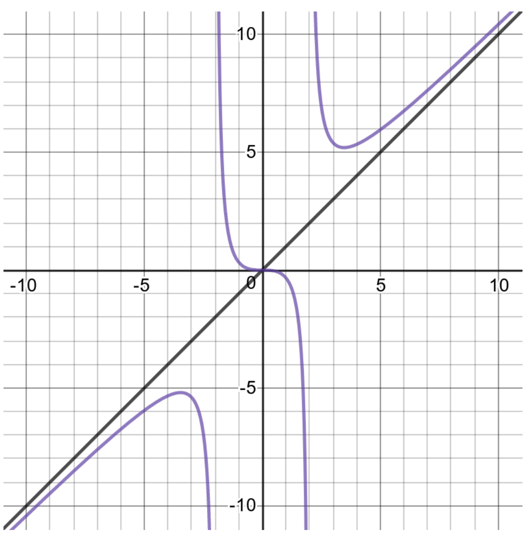
So f' can change sign only when $x = \pm 2$, $x = 0$, or $x = \pm\sqrt{12}$. The denominator is always positive, so we only need to check when the numerator is positive. Moreover, $f'(x)$ is even, so we just need to look at what happens when $x > 0$. From this we see that $f'(x) > 0$ when $|x| > \sqrt{12}$, and $f'(x) < 0$ otherwise.

I would compute f'' as follows, but of course there are several ways you can do it, and they're all fine.

$$
\begin{aligned}
f'(x) &= \left(1 + \frac{4}{x^2-4} - \frac{8x^2}{(x^2-4)^2}\right)' \\
&= \frac{-8x}{(x^2-4)^2} - \frac{16x}{(x^2-4)^2} + \frac{32x^3}{(x^2-4)^3} \\
&= \frac{-8x(x^2-4) - 16x(x^2-4) + 32x^3}{(x^2-4)^3} \\
&= \frac{-8x^3 + 32x - 16x^3 + 64x + 32x^3}{(x^2-4)^3} \\
&= \frac{8x^3 + 96x}{(x^2-4)^3} \\
&= \frac{8x(x^2+12)}{(x^2-4)^3}
\end{aligned}
$$

Then we can see that $f''(x)$ can change sign only at $x = \pm 2$ or $x = 0$. When x crosses -2 or 2 then the denominator changes sign, and when x crosses 0 the numerator changes sign, so $f''(x) < 0$ when $x < -2$ and $0 < x < 2$ and $f''(x) > 0$ otherwise.

Finally note that f has vertical asymptotes at $x = \pm 2$, and is asymptotic to x when x is very large or very small. Combining this with the positive/negative, increasing/decreasing, and concave up/concave down data we got from examining the derivatives, we can produce this sketch (with $y = x$ as a guide):



2. There are a lot of different ways to solve this problem. Here's what first comes to mind for me, but if you prefer your method that's completely fine. The most important thing to get from this problem is the attitude that you should just try things and see what works.

a) This is a cubic polynomial. Let's call it f. Then $f(x) = ax^3 + bx^2 + cx + d$. Because $(0, 3)$ is on the graph we can conclude that $d = 3$. Moreover, $f(x) - 3$ is an odd function, so we know that $b = 0$. There's a minimum at $x = 1$, so $f'(1) = 0$. We compute that $f'(x) = 3ax^2 + c$ (using the fact that $b = 0$), and then $f'(1) = 0$ gives $3a + c = 0$, or $c = -3a$. Then we can use the fact that $f(1) = 1$ to get $a \cdot 1^3 - 3a \cdot 1 + 3 = 1$, giving

$a = 1$. Thus our polynomial is $x^3 - 3x + 3$. We could also have found that $b = 0$ using the fact that $f''(0) = 0$, since the graph has an inflection point at $x = 0$, or using $f(1) + f(-1) = 6$. We could have also found everything by just using the four equations $f(0) = 3$, $f(1) = 1$, $f(-1) = 5$, and $f(-2) = 1$. There are a lot of options.

b) This is a quartic (degree 4) polynomial. Let's call it g. We see that g has a root at $x = 1$, and moreover $x = 1$ is what's called a *double root*: g's derivative there is also 0. Write $g(x) = p(x)(x-1)$. Here p is some unknown polynomial. Using the product rule, $g'(x) = p'(x)(x-1) + p(x)$. Then using the fact that $g'(1) = 0$ we see that $p(1) = 0$, i.e. $p(x) = (x-1)q(x)$ for some polynomial q. Thus $g(x) = (x-1)^2 q(x)$. Similarly $x = -1$ is a double root, so $g(x)$ has a factor of $(x+1)^2$. This already tells us that $g(x) = a(x-1)^2(x+1)^2$ for some real number a. Then we can observe that $g(1) = 1$ to find that $a = 1$.

c) This is a quntic polynomial. Call it $h(x)$. It's odd, so we can write $h(x) = ax^5 + bx^3 + cx$. We see that $(4, h(4))$ is a local minimum of h, so $5 \cdot 4^4 a + 3 \cdot 4^2 b + c = 0$. Now we have to squint a bit to proceed. There are two options. The first is to notice that h has an inflection point at $x = 3$. It's not totally clear exactly where the inflection point is, but it's definitely close to 3, and I tell you in the problem statement that I made these x-coordinates simple, and with that clue you can be pretty confident that $h''(3) = 0$. This gives us the equation $20 \cdot 3^3 a + 6 \cdot 3b = 0$. From this equation we get that $b = -30a$, and then the equation $5 \cdot 4^4 a + 3 \cdot 4^2 b + c = 0$ becomes $c = 160a$. So far we've found that $h(x) = a(x^5 - 30x^3 + 160x)$. To find a we need to look at some y-value. Otherwise we'd never be able to distinguish between h and a multiple of h. We don't have any clear y-values, but $h(4)$ is between -200 and -300, and if we evaluate $h(4)$ using the expression for h we have, we get $h(4) = -256a$, so a is probably 1. Extra squinting confirms this.

Instead of using the inflection point at $x = 3$ we could also have used the maximum around $x = 1.4$. I said in the question that I made the x-coordinates "as simple as possible" so if this x-coodrinate were actually 1.4, or some other rational number, I probably would have just given you $5^5 h\left(\frac{1}{5}x\right)$ instead of $h(x)$ so that this x-coordinate was an integer and the polynomial still had integer coefficients. Using this reasoning (for example; other lines of reasoning work too) you can guess that this x-value is not rational. Since I tell you that it's a simple number, from here it's not hard to identify it at $\sqrt{2}$. In general it's very helpful to be aware that a number you're squinting at might be irrational, and there are a lot of specific irrational numbers that come up over and over again, like $\sqrt{2}$. From here, you can use $h'(\sqrt{2}) = 0$ to solve for a, b, and c as above.