Calculus 1 Assignment 9 Solutions

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1.

a) We can split the interval $[0,1]$ into 4 equal parts: $\left[\frac{0}{4},\frac{1}{4}\right],\left[\frac{1}{4},\frac{2}{4}\right],\left[\frac{2}{4},\frac{3}{4}\right]$, and $\left[\frac{3}{4},\frac{4}{4}\right]$. For each of these parts, we can estimate the area under the curve $y=x^2$ with a rectangle with base equal to the length of the interval, and height equal to the value of the function $f(x)=x^2$ evaluated at the right endpoint. The areas of these four rectanges are $\frac{1}{4}\cdot\left(\frac{1}{4}\right)^2, \frac{1}{4}\cdot\left(\frac{2}{4}\right)^2, \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2$, and $\frac{1}{4}\cdot\left(\frac{4}{4}\right)^2$. Our estimate of the total area is then

$$\frac{1}{4}\left(\frac{1}{4}\right)^2+\frac{1}{4}\left(\frac{2}{4}\right)^2+\frac{1}{4}\left(\frac{3}{4}\right)^2+\frac{1}{4}\left(\frac{4}{4}\right)^2=\frac{1^2+2^2+3^2+4^2}{4^3}.$$

b) Using the same idea as above, but with n rectangles instead of 4, we'll obtain the sum

$$\frac{1}{n^3}\sum_{k=1}^{n}k^2.$$

A quick google search tells us that this expression is equal to $\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}$. To obtain the area exactly, we should then take the limit as n goes to infinity of this approximate area. This limit is equal to $\frac{1}{3}$.

2.

a) We can split the interval $[0,1]$ into four intervals, like we did in problem 1a). For each of these intervals $[x_\ell, x_r]$, we can use the same idea for approximation as given in the problem statement, which is to approximate the length of the curve as the distance between the "starting point" (x_ℓ, x_ℓ^2) and the "ending point" (x_r, x_r^2). This distance is equal to $\sqrt{(x_r-x_\ell)^2+(x_r^2-x_\ell^2)^2}$. We should then approximate the total length of our curve as being the sum of these four distances. If our intervals are $\left[\frac{0}{4},\frac{1}{4}\right],\left[\frac{1}{4},\frac{2}{4}\right],\left[\frac{2}{4},\frac{3}{4}\right]$, and $\left[\frac{3}{4},\frac{4}{4}\right]$, then this comes out to

$$\sqrt{\left(\frac{1}{4}-\frac{0}{4}\right)^2+\left(\left(\frac{1}{4}\right)^2-\left(\frac{0^2}{4}\right)\right)^2}+\sqrt{\left(\frac{2}{4}-\frac{1}{4}\right)^2+\left(\left(\frac{2}{4}\right)^2-\left(\frac{1}{4}\right)^2\right)^2}$$
$$+\sqrt{\left(\frac{3}{4}-\frac{2}{4}\right)^2+\left(\left(\frac{3}{4}\right)^2-\left(\frac{2}{4}\right)^2\right)^2}+\sqrt{\left(\frac{4}{4}-\frac{3}{4}\right)^2+\left(\left(\frac{4}{4}\right)^2-\left(\frac{3}{4}\right)^2\right)^2}$$
$$=\sum_{k=1}^{4}\sqrt{\left(\frac{1}{4}\right)^2+\left(\left(\frac{k}{4}\right)^2-\left(\frac{k-1}{4}\right)^2\right)^2}.$$

b) Using the same reasoning as above, but with n intervals instead of 4, gives the estimate

$$\sum_{k=1}^{n}\sqrt{\left(\frac{1}{n}\right)^2+\left(\left(\frac{k}{n}\right)^2-\left(\frac{k-1}{n}\right)^2\right)^2}$$

for the length of the curve. Taking the limit as n goes to infinity then gives the length of the curve exactly.

If we think of the individual terms as being of the form $\sqrt{(\Delta x)^2 + (\Delta y)^2}$, then maybe we'll have the idea of factoring out a Δx to get $\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$. Then as we take the limit, it can be verified that $\frac{\Delta y}{\Delta x}$ becomes the derivative $\frac{dy}{dx}$, so people usually instead write that the length of the curve is

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sqrt{1 + f'\left(\frac{k}{n}\right)^2},$$

where here $f(x) = x^2$.

3. The choice doesn't matter in the limit. We split the interval of time $[0, 3\mathrm{h}]$ into n intervals of equal length. When we take the right endpoints of those intervals to sample our velocity function $v(t)$, we get that our estimate for the total distance travelled is

$$\sum_{k=1}^{n} \frac{3\mathrm{h}}{n} v\left(k\frac{3\mathrm{h}}{n}\right),$$

and if we use left endpoints instead, then our estimate is

$$\sum_{k=0}^{n-1} \frac{3\mathrm{h}}{n} v\left(k\frac{3\mathrm{h}}{n}\right).$$

The difference between these two estimates is

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{3\mathrm{h}}{n} v\left(k\frac{3\mathrm{h}}{n}\right) - \sum_{k=0}^{n-1} \frac{3\mathrm{h}}{n} v\left(k\frac{3\mathrm{h}}{n}\right)\\
&= \left[\frac{3\mathrm{h}}{n} v\left(1\cdot\frac{3\mathrm{h}}{n}\right) + \frac{3\mathrm{h}}{n} v\left(2\cdot\frac{3\mathrm{h}}{n}\right) + \frac{3\mathrm{h}}{n} v\left(3\cdot\frac{3\mathrm{h}}{n}\right) + \cdots + \frac{3\mathrm{h}}{n} v\left((n-1)\cdot\frac{3\mathrm{h}}{n}\right) + \frac{3\mathrm{h}}{n} v\left(n\cdot\frac{3\mathrm{h}}{n}\right)\right]\\
&\qquad - \left[\frac{3\mathrm{h}}{n} v\left(0\cdot\frac{3\mathrm{h}}{n}\right) + \frac{3\mathrm{h}}{n} v\left(1\cdot\frac{3\mathrm{h}}{n}\right) + \frac{3\mathrm{h}}{n} v\left(2\cdot\frac{3\mathrm{h}}{n}\right) + \cdots + \frac{3\mathrm{h}}{n} v\left((n-1)\cdot\frac{3\mathrm{h}}{n}\right)\right]\\
&= \frac{3\mathrm{h}}{n} v\left(n\cdot\frac{3\mathrm{h}}{n}\right) - \frac{3\mathrm{h}}{n} v\left(0\cdot\frac{3\mathrm{h}}{n}\right)
\end{aligned}$$

(all the middle terms appear in both sums and thus disappear when taking the difference). This quantity goes to 0 as n goes to infinity. This phenomenon can also be observed by drawing rectangles. If you draw n rectangles, the difference between their areas is given by n small rectangles on the diagonal of area $\mathcal{O}(n^{-2})$. Thus the difference in area is $\mathcal{O}(n^{-1})$.